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The same principle is our guide in the placing of the products here. For the first product we have $3+4=7$, and for the second $3+6=9$, the latter falling *two* places in advance of the former, in consequence of the occurrence of a cipher. For the next we have $12-7=5$; that is, the fifth decimal place of the second multiplicand is the last effective figure, its product falling in the twelfth place. And here, also, the operation changes into that of contracted multiplication.

It is with a view to perspicuity that I have exemplified the three processes, as each of the last two follows from that which precedes it by a natural sequence. But it will be understood that it is the tables for the application of the three-figure process alone that accompany these papers. The superiority of the process last named over the others, in regard to the number of tabular entries and of figures employed, is equally apparent in the anti-logarithmic as in the logarithmic application. It will be still more fully appreciated after the trial of a few examples.

I must postpone till another opportunity a description of the methods employed for the construction and verification of the tables, and also an account of the origin of the processes that have been exemplified.

On Interpolation, Summation, and the Adjustment of Numerical Tables. (Part III.) By W. S. B. Woolhouse, F.R.A.S., F.S.S., *Vice-President of the Institute of Actuaries, &c.*

[Read before the Institute, 27th February, 1865.]

THE ADJUSTMENT OF NUMERICAL TABLES.

WHEN a series of quantities which depend on a fixed law, whether known or implied, follow each other in a due order of succession, the general accuracy of their numerical values may be satisfactorily tested by observing the regularity of the progression of a suitable order of differences. If the tabular quantities are the results of calculation from a given formula, with equidistant arguments, by differencing them up to a certain order the existence of an isolated error, if one should exist, is thus prominently exhibited and therefore speedily detected. Moreover the precise locality of the error and the fact of its individuality being indicated by a characteristic interruption of the law of progression, the proper correction is ascertained by carefully noting the central position of the disturbance, and revising the corresponding portion of the original computation. This work of revision may, however, be dispensed with, if necessary or desirable, as the magnitude as well as the locality of the indicated error can in general be readily deduced, with sufficient accuracy, directly from the differences, by the method of adjustment here given. The method, being so far independent of fundamental origin, is therefore equally applicable to the adjustment of a series of quantities derived from observation, or from statistical data; only in this latter case, as almost every quantity must be more or less affected by imperfections arising from defective information and other known or unknown incidental causes, the influence of which may sometimes be augmented by paucity of numbers, the progression of the differences is disturbed by a more complicated combination of the effects of an irregular series of errors, and it becomes more difficult to eliminate the value of each separate correction.

In constructing and adjusting numerical tables, more especially those drawn from statistics, the method commonly employed is simply that of taking as data only a few of the unadjusted tabular values, the same being distributed apart at certain assigned convenient intervals, and thence determining all the intermediate values by a suitable process of interpolation, so as to impart to the results a desirable graduated progression. Although expedient as regards facility of calculation, this practice is in other respects devoid of scientific principle. The meagre data upon which everything is made to depend may indeed consist of values amongst those most in error, and under any circumstances an undue and exclusive presumption of accuracy is tacitly given to a few values arbitrarily chosen to the exclusion of all the others; and by using these values without any preliminary adjustment, the entire table is necessarily and systematically vitiated by whatever imperfections they may happen to possess. The real problem to be solved is to determine a method of constructing a properly graduated table, which shall represent the whole of a given series of values, or a given mass of original facts, with a greater degree of accuracy than that of any other table of a similar kind. A thorough investigation of the subject, thus considered, and taken in all its bearings, may be said to open out an entirely new branch of science; and in the discussion given in the present paper it will, I think, be admitted that some progress has been made towards its mathematical development.

The uniform interval between the consecutive arguments of the table being supposed to be sufficiently small, it will follow that if the exact functions were differenced, the differences would rapidly converge and exhibit no sign of discrepancy. But the given functions or quantities involve, in addition to the exact functions, their respective errors; and it is evident that the amounts of variation of the several differences would be produced by separately differencing the series of errors which, on account of their irregularity and consequent divergence of the differences, soon become prominently conspicuous.

ISOLATED ERRORS.

First consider the case of a single error. Let the value V_0 be in error so as to require a correction v, and let the values which immediately precede and follow it be supposed to be exact. Then the effects produced on the several orders of differences by the correction v will be as follows:—

Correction.	Δ_1	Δ_2	Δ_3	Δ_4	Δ_5	Δ_6	Δ_7	Δ_8
0		- -		- -		- -		$+v$
	- -		- -		- -		$+v$	
0		- -		- -		$+v$		$-8v$
	- -		- -		$+v$		$-7v$	
0		- -		$+v$		$-6v$		$+28v$
	- -		$+v$		$-5v$		$+21v$	
0		$+v$		$-4v$		$+15v$		$-56v$
	$+v$		$-3v$		$+10v$		$-35v$	
$\ldots v$		$-2v$		$+6v$		$-20v$		$+70v$
	$-v$		$+3v$		$-10v$		$+35v$	
0		$+v$		$-4v$		$+15v$		$-56v$
	- -		$-v$		$+5v$		$-21v$	
0		- -		$+v$		$-6v$		$+28v$
	- -		- -		$-v$		$+7v$	
0		- -		- -		$+v$		$-8v$
	- -		- -		- -		$-v$	
0		- -		- -		- -		$+v$

These are simply deduced by successive differencing in an expanded form, the absence of a symbol and the insertion of hyphens being always accounted as zero; and it will be observed that the numerical coefficients which appertain to each order n are those of the expansion of the binomial $(1-1)^n$.

Retaining the notation before employed ("Interpolation," vol. xi., page 64), the corresponding portion of the differences of the given series of quantities will stand thus:—

Given Values.	Δ_1	Δ_2	Δ_3	Δ_4	Δ_5	Δ_6	Δ_7	Δ_8
V_{-4}								h_{-4}
							g_{-4}	
V_{-3}						f_{-3}		h_{-3}
					e_{-3}		g_{-3}	
V_{-2}				d_{-2}		f_{-2}		h_{-2}
			c_{-2}		e_{-2}		g_{-2}	
V_{-1}		b_{-1}		d_{-1}		f_{-1}		h_{-1}
	a_{-1}		c_{-1}		e_{-1}		g_{-1}	
$\ldots V_0$		b_0		d_0		f_0		h_0
	a_{+1}		c_{+1}		e_{+1}		g_{+1}	
V_{+1}		b_{+1}		d_{+1}		f_{+1}		h_{+1}
			c_{+2}		e_{+2}		g_{+2}	
V_{+2}				d_{+2}		f_{+2}		h_{+2}
					e_{+3}		g_{+3}	
V_{+3}						f_{+3}		h_{+3}
							g_{+4}	
V_{+4}								h_{+4}

Now the quantity V_0, when corrected, will be V_0+v; and the adjustments produced by v on these differences will in like manner be obtained by incorporating the respective values contained in the former table.

To detect the existence, as well as to determine the value of the correction v, it will be most convenient to have recourse to an even order of differences, as the greatest disturbance is then exhibited in the central difference, or that which stands opposite to V_0, the quantity in error. The practical methods of calculation which follow will now be obvious.

DETERMINATION OF AN ERROR FROM SECOND DIFFERENCES.

An error in the value of a quantity V_0 will be indicated by a marked disturbance of three second differences, b_{-1}, b_0, b_{+1}, and the required correction is to be inferred so as to reduce them to an arithmetical progression. Thus, when the corrected second differences $b_{-1}+v$, b_0-2v, $b_{+1}+v$, are in arithmetical progression, the two third differences, $c_{-1}-3v$, $c_{+1}+3v$, derived from them, will be equal, and consequently the fourth difference, d_0+6v, will vanish or become neutralized;

$$\therefore \text{ the correction } (v) = -\frac{d_0}{6} = \frac{(b_0-b_{-1})+(b_0-b_{+1})}{6}.$$

Hence the following simple rule, which may in general be applied mentally or by mere inspection, it being, of course, always understood that proper attention is given to the algebraic signs + and −.

RULE.—*From the middle second difference subtract separately the preceding and following differences; add together the two remainders, and divide by 6. The result, with its proper algebraic sign, will be the correction to be made in the value which stands opposite to the middle difference.*

Example.—To test the accuracy of a consecutive series of tabular numbers, they are differenced as follows:—

Tabular Numbers.	First Differences.	Second Differences.	Corrections of Second Differences.	Adjusted Second Differences.
1728				
	+469			
2197		+ 78		
	547			
2744		84		
	631			
3375		103*	$-13 \ldots v$	+90
	734			
*4109		70*	$+26 \ldots -2v$	96
	804			
4913		115*	$-13 \ldots v$	+102
	919			
5832		108		
	1027			
6859		+114		
	+1141			
8000		&c.		
	&c.			
&c.				

Here the three second differences distinguished by an asterisk are decidedly irregular as regards progression, and show that the tabular number opposite to the central difference (+70) must be in error. And for the adjustment of this error, according to the rule, we have

$$\begin{array}{l} \text{Irregular} \\ \text{Second} \\ \text{Differences} \end{array} \left\{ \begin{array}{r} +103 \\ 70 \\ +115 \end{array} \right. \quad \begin{array}{rl} -33 \ldots\ldots & \text{middle}-\text{preceding diff.} \\ -45 \ldots\ldots & \text{middle}-\text{following diff.} \\ \hline 6)-78 & \end{array}$$

Correction v.... -13

V 4109

Adjusted value 4096

By replacing 4109 by this adjusted value, and revising the differences, they will now observe a regular progression, showing that no other error exists amongst the numbers put down. The second differences may, however, be otherwise directly and independently adjusted by applying to their original values the respective corrections $+v$, $-2v$, $+v$, which is already done at the right hand of the above table.

The tabular numbers quoted in this example are the cubes of the numbers 12, 13, 14, ... 20. By differencing them out to fourth differences, even a single unit in error is at once disclosed; thus:—

TRUE VALUES.				
Numbers.	Differences.			
1728				
	+ 469			
2197		+ 78		
	547		+6	
2744		84		0
	631		6	
3375		90		0
	721		6	
4096		96		0
	817		6	
4913		102		0
	919		6	
5832		108		0
	1027		+6	
6859		+114		
	+1141			
8000				

PRIMARY ERROR OF AN UNIT.				
Numbers.	Differences.			
1728				
	+ 469			
2197		+ 78		
	547		+6	
2744		84		+1
	631		7	
3375		91		−4
	722		3	
*4097		94		+6
	816		9	
4913		103		−4
	919		5	
5832		108		+1
	1027		+6	
6859		+114		
	+1141			
8000				

The differences $(+1, -4, +6, -4, +1)$ here agree in character with the fourth difference column of the first table, and thereby show that the indicated error is the only one that exists.

DETERMINATION OF AN ERROR FROM FOURTH DIFFERENCES.

The value of the correction of a quantity may in like manner be deduced from three disturbed fourth differences, so as to fulfil the condition that the three differences when adjusted shall form an arithmetical progression. The corrected fourth differences are here $d_{-1}-4v$, d_0+6v, $d_{+1}-4v$; and when these are in arithmetical progression, the two fifth differences, $e_{-1}+10v$, $e_{+1}-10v$, derived from them, will be equal, and therefore the sixth difference, f_0-20v, will vanish or become neutralized;

$$\therefore \text{ the correction } (v)=\frac{f_0}{20}=\frac{(d_{-1}-d_0)+(d_{+1}-d_0)}{20},$$

which suggests the following rule:—

RULE.—*From the preceding and following fourth differences separately subtract the middle difference; add together the two remainders, and divide by* 20. *The quotient, with its proper algebraic sign, will be the required correction, and will be applicable to the quantity which stands opposite to the middle difference.*

By the foregoing table, showing the effects produced by v on the differences, it will appear that five fourth differences will be subjected to alteration, and that they may be adjusted, with respect to v, without redifferencing, by directly applying to them the several corrections $+v$, $-4v$, $+6v$, $-4v$, $+v$. If the differences so corrected, when taken in connexion with those that immediately precede and follow them, should then be found to observe a satisfactory progression, the fact will show that the primitive values immediately adjoining V_0 are free from material error. But if any marked irregularities should yet remain, other corrections will require to be made in a similar manner; and by thus continuing to repeat the process at points of greatest derangement, all the more prominent disturbing inequalities of the given series of numbers may be ultimately regulated.

Whether second or fourth differences should be used in these calculations for the adjustment of any proposed set of numbers, must, to a certain extent, be left to the practical judgment of the computer. Perhaps fourth differences will be found to be the more generally applicable. As a general guide, however, it may be advisable first to consider what even order of differences of exact functions ought to be small enough to be practically disregarded, and then to employ differences of the next less even order. And here it will not pass unnoticed that the foregoing simple rules of calculation may, if required, be readily adapted to higher orders of differences.

The rules, as they are here given, were, many years ago, uniformly and successfully employed by myself in officially examining and testing the accuracy of the results of the various and extensive computations made in the Nautical Almanac Office; and I also availed myself of their application in the adjustment of both the "Indian Military" and the "Experience" tables of mortality.* In the construction of all kinds of tables these rules will in general be found to be very effective in the adjustment of isolated or prominent irregularities, as they proceed upon the sound principle of applying each successive correction at the very point where the probable or certain existence of an error is most strongly indicated. If, for example, a number of errors, each of which may be of any magnitude, were promiscuously introduced into a page of a table of logarithms, or in any table computed mathematically according to a given formula or generic law which is still inherent to the general mass of values, the preceding rules, when applied to the differences, would necessarily have the effect of practically eliminating the several errors and restoring the table to its original state of accuracy. In all ordinary calculations involving the construction and adjustment of tables what has already been laid down may therefore be considered to meet every necessary requirement.

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* "*Investigation of Mortality in the Indian Army.* By W. S. B. Woolhouse, F.R.A.S., &c.," and "*Tables Exhibiting the Law of Mortality deduced from the Combined Experience of Seventeen Life Assurance Offices.*" It is presumed that no existing graduated table of mortality represents so closely the general body of facts on which it is founded as the tables here referred to. Mr. Jellicoe, the President of the Institute, at the end of a valuable paper inserted in vol. i., page 166, has given a D and N table, together with the values of an annuity and annual premium, calculated from the adjusted Bengal Military Mortality, at 4 per cent.

MULTIFORM ERRORS.

We have yet, however, to enter upon a discussion of another part of the subject, which, if it be less essential in ordinary practice, nevertheless admits of most important applications, and also possesses considerable interest in a mathematical point of view. If the generic law be unknown, and the table—as in the case of a table of mortality—be drawn from observation, then no part of it can be predicated as really or even probably accurate, and therefore, after the more prominent displacements and straggling errors have been adjusted as before described, it may afterwards be requisite, if a continuous and accurately-finished table be desired, to have recourse to some other more recondite process in order to finally eradicate a consecutive series of lesser imperfections, the disturbing effects of which on the differences are of course blended together without exhibiting any individual traces.

To proceed to an investigation of the subject in this more complicated form, we have now to discuss the question of the complete adjustment of a combination of multiform small errors, or the neutralization of the assigned discrepancies of an irregular series of differences of a given order.

In the first place we shall consider the case in relation to fourth differences. Let the given original quantities be denoted by $V_0, V_1, V_2, \dots . V_n$, and the sought corrections, with their proper algebraic signs, respectively by $v_0, v_1, v_2, \dots . v_n$; and suppose them to be separately differenced as follows:—

$$
\begin{array}{c|c|c|c|c}
V_0 &&&& \\
& a_0 &&& \\
V_1 && b_0 && \\
& a_1 && c_0 & \\
V_2 && b_1 && d_0 \\
& a_2 && c_1 & \\
V_3 && b_2 && d_1 \\
& a_3 && c_2 & \vdots \\
V_4 && b_3 & \vdots & \vdots \\
& a_4 & \vdots & \vdots & \\
V_5 & \vdots & \vdots && d_{n-4} \\
\vdots & \vdots && c_{n-3} & \\
\vdots && b_{n-2} && \\
& a_{n-1} &&& \\
V_n &&&&
\end{array}
\qquad\qquad
\begin{array}{c|c|c|c|c}
v_0 &&&& \\
& \alpha_0 &&& \\
v_1 && \beta_0 && \\
& \alpha_1 && \gamma_0 & \\
v_2 && \beta_1 && \delta d_0 \\
& \alpha_2 && \gamma_1 & \\
v_3 && \beta_2 && \delta d_1 \\
& \alpha_3 && \gamma_2 & \vdots \\
v_4 && \beta_3 & \vdots & \vdots \\
& \alpha_4 & \vdots & \vdots & \\
v_5 & \vdots & \vdots && \delta d_{n-4} \\
\vdots & \vdots && \gamma_{n-3} & \\
\vdots && \beta_{n-2} && \\
& \alpha_{n-1} &&& \\
v_n &&&&
\end{array}
$$

Then $V_0+v_0, V_1+v_1, \dots . V_n+v_n$ are the corrected functions, and their corresponding fourth differences will evidently be $d_0+\delta d_0, d_1+\delta d_1, \dots . d_{m-4}+\delta d_{m-4}$, where δ is employed as a characteristic of variation, and not as a quantity. In most cases it may be practicable to assign such small values to $\delta d_0, \delta d_1, \dots . \delta d_{n-4}$ as shall impart a tolerable progression to these corrected differences, and thus neutralize the principal irregularities of the given differences $d_0, d_1, \ldots . d_{n-4}$. The values δd_0, δd_1, &c., thus found, will be the only data for the determination of $v_0, v_1, \ldots . v_n$; and as the number of these data, or mathematical conditions, is here less by four than the number of unknown values, the problem remains so far essentially indeterminate. This is indeed evident from the consideration that by arbitrarily assuming any four numbers for the unknown leading differences v_0, α_0, β_0, γ_0, and using these numbers in combination with the assigned fourth differences $\delta d_0, \delta d_1, \ldots . \delta d_{n-4}$, the several values of v, in common with the whole scheme of differences, become directly deducible by summation. The gist of the present inquiry is to assign such values to v_0, α_0, β_0, γ_0, that the resulting displacements, $v_0, v_1, \ldots v_n$, of the original functions, shall be as small as possible. To effect this object the general amount of disturbance is most satisfactorily estimated by adding together the squares of the several corrections, or errors, for by that means positive and negative disturbances, which are alike objectionable, become all of them positive, and therefore accumulative. Hence the problem, mathematically enunciated, resolves itself into the following :—

PROBLEM.—*To find a series of small quantities, $v_0, v_1, \ldots . v_n$, which shall have a given set of fourth differences, and such that the sum of their squares shall be a minimum.*

Before proceeding with the investigation of this problem, it may not be out of place here to premise a brief outline of the practical resolution of equations of condition by Gauss's "method of least squares," as the same may not be generally known or understood. At least, I believe it has not before been explained on elementary algebraical principles.

Let a series of errors be expressed by the linear equations

$$\begin{aligned} e_0 &= a_0x + b_0y + c_0z \ \ \&c. \\ e_1 &= a_1x + b_1y + c_1z \ \ \&c. \\ e_2 &= a_2x + b_2y + c_2z \ \ \&c. \\ &\&c. \qquad \&c. \end{aligned}$$

where a_0, b_0, c_0, &c. are given coefficients; let the number of equations exceed the number of variables x, y, z, &c.; and let it be required to find values of these variables, such that the sum of the squares of the errors, or

$$S = e_0^2 + e_1^2 + e_2^2 + \&c.$$

shall be a minimum.

When this last condition subsists, any small alteration, whether positive or negative, in the values of one or more of the variables, must increase the value of S. Suppose x to become $x+\varepsilon$; then the series of errors will evidently become $e_0+a_0\varepsilon$, $e_1+a_1\varepsilon$, $e_2+a_2\varepsilon$, &c., and therefore the sum of their squares will be

$$S+2(a_0e_0+a_1e_1+a_2e_2\ldots.)\varepsilon+(a_0^2+a_1^2+a_2^2\ldots.)\varepsilon^2.$$

Here the third term, involving ε^2, being essentially positive, it is evident that the value of the expression will always exceed S, whether ε be positive or negative, provided that

$$a_0e_0+a_1e_1+a_2e_2 \text{ \&c.}=0.$$

And in a similar manner, by separately varying y, z, &c., it may be inferred that the same condition will be further maintained when

$$\begin{array}{l} b_0e_0+b_1e_1+b_2e_2 \text{ \&c.}=0 \\ c_0e_0+c_1e_1+c_2e_2 \text{ \&c.}=0 \\ \quad\text{\&c.} \qquad\quad \text{\&c.} \end{array}$$

If in each of these equations the errors e_0, e_1, e_2, &c., be replaced respectively by the preceding expressions, they will serve for the determination of the variables. Hence the following rule:—

RULE.—*Multiply the equations severally by the respective coefficients of one particular variable, and equate the sum of the products with zero. By proceeding thus with respect to each separate variable, as many equations as variables are obtained; and by solving these equations, of the first degree, the values of the several variables are ultimately determined.*

To apply this method of least squares to the present inquiry it will be requisite to express the general value (v_m) in terms of v_0, a_0, β_0, γ_0, and the given fourth differences $\delta d_0, \delta d_1, \ldots. \delta d_{n-4}$. This will be readily effected by summation. Thus, we have evidently

$$\begin{array}{rl} \gamma_0 & =\gamma_0 \\ \gamma_1 & =\gamma_0+\delta d_0 \\ \gamma_2 & =\gamma_0+\delta d_0+\delta d_1 \\ \gamma_3 & =\gamma_0+\delta d_0+\delta d_1+\delta d_2 \\ \vdots & \\ \gamma_{m-3} & =\gamma_0+\delta d_0+\delta d_1+\delta d_2\ldots.+\delta d_{m-4} \\ \therefore \beta_{m-2} & =\beta_0+\Sigma\gamma \\ & =\beta_0+(m-2)\gamma_0+(m-3)\delta d_0+(m-4)\delta d_1 \text{ \&c.} \\ & =\beta_0+(m-2)\gamma_0+\Sigma_x\{(m-\overline{x+3})\delta d_x\}. \end{array}$$

And, bearing in mind, the usual formula for the summation of a common factorial, viz.,

$$\Sigma_{1\ldots\ldots n}p(p+1)\ldots\ldots(p+n)=\frac{p(p+1)\ldots\ldots(p+n+1)}{n+2}\quad\ldots(f)$$

we similarly find

$$\begin{aligned}\alpha_{m-1}&=\alpha_0+\Sigma\beta\\&=\alpha_0+(m-1)\beta_0+\frac{(m-2)(m-1)}{2}\gamma_0\\&\qquad+\Sigma\frac{(m-\overline{x+3})(m-\overline{x+2})}{2}\delta d_x\,;\end{aligned}$$

and $\therefore\ v_m=v_0+\Sigma\alpha$

$$\begin{aligned}&=v_0+m\alpha_0+\frac{(m-1)m}{2}\beta_0+\frac{(m-2)(m-1)m}{2.3}\gamma_0\\&\qquad+\Sigma\frac{(m-\overline{x+3})(m-\overline{x+2})(m-\overline{x+1})}{2.3}\delta d_x\quad\ldots(\delta)\end{aligned}$$

where Σ indicates a summation with respect to x from 0 to $m-4$.

When m takes the consecutive values 0, 1, 2 . . . n, the formula (δ) will express with mathematical accuracy the several values of the series of corrections, and as the coefficients of the four variables v_0, α_0, β_0, γ_0, are respectively 1, m, $\frac{m(m-1)}{2}$, $\frac{m(m-1)(m-2)}{2.3}$, the resolving equations, by the method of least squares, so as to make Σv^2 a minimum, will be, according to the foregoing rule,

$$\Sigma v_m=0,\quad \Sigma m v_m=0,\quad \Sigma\frac{m(m-1)}{2}v_m=0,\quad \Sigma\frac{m(m-1)(m-2)}{2.3}v_m=0;$$

and these are evidently equivalent to

$$\Sigma v_m=0,\quad \Sigma m v_m=0,\quad \Sigma m^2 v_m=0,\quad \Sigma m^3 v_m=0\quad\ldots\ldots(e).$$

In effecting these summations with respect to m, from 0 to n, after substituting the formula (δ), the term involving δd_x will, in each case, consist of factorials, by further substituting the algebraical equivalents,

$$\begin{aligned}m\ &=(m-x)+x\\m^2&=(m-x)(m-\overline{x-1})+(2x-1).(m-x)+x^2\\m^3&=(m-x)(m-\overline{x-1})(m-\overline{x-2})+3(x-1).(m-x)(m-\overline{x-1})\\&\qquad+(3c^2-3x+1).(m-x)+x^3.\end{aligned}$$

To abbreviate the process, let

$$\mu_x=\frac{(n-\overline{x+3})(n-\overline{x+2})(n-\overline{x+1})(n-x)}{2.3}\delta d_x\quad\ldots(g);$$

also,

$$
\begin{aligned}
\mathrm{M} &= \Sigma\mu_x \\
\mathrm{P} &= \Sigma\mu_x\{4n+(x+4)\} \\
\mathrm{Q} &= \Sigma\mu_x\{10n^2+(4x+18)n+(x^2+6x+8)\} \\
\mathrm{R} &= \Sigma\mu_x\{20n^3+(10x+50)n^2+(4x^2+26x+38)n \\
&\qquad +(x^3+8x^2+18x+8)\}.
\end{aligned}
$$

Then, for all consecutive values of m from 0 to n, the results of the respective summations contained in (e) will be

$$
\begin{aligned}
&(n+1)v_0+\frac{(n+1)n}{2}\alpha_0+\frac{(n+1)n(n-1)}{2.3}\beta_0 \\
&\qquad+\frac{(n+1)n(n-1)(n-2)}{2.3.4}\gamma_0+\frac{\mathrm{M}}{4}=0, \\
&\frac{(n+1)n}{2}v_0+\frac{(n+1)n(2n+1)}{2.3}\alpha_0+\frac{(n+1)n(n-1)(3n+2)}{2.3.4}\beta_0 \\
&\qquad+\frac{(n+1)n(n-1)(n-2)(4n+3)}{2.3.4.5}\gamma_0+\frac{\mathrm{P}}{4.5}=0, \\
&\frac{(n+1)n(2n+1)}{2.3}v_0+\frac{(n+1)^2n^2}{4}\alpha_0+\frac{(n+1)n(n-1)(12n^2+15n+2)}{2.3.4.5}\beta_0 \\
&\qquad+\frac{(n+1)n(n-1)(n-2)(10n^2+14n+3)}{3.4.5.6}\gamma_0+\frac{\mathrm{Q}}{3.4.5}=0, \\
&\frac{(n+1)^2n^2}{4}v_0+\frac{(n+1)n(2n+1)(3n^2+3n-1)}{2.3.5}\alpha_0 \\
&\qquad+\frac{(n+1)n(n-1)(10n^3+18n^2+5n-2)}{2.3.4.5}\beta_0 \\
&\qquad+\frac{(n+1)n(n-1)(n-2)(20n^3+40n^2+16n-3)}{4.5.6.7}\gamma_0+\frac{\mathrm{R}}{4.5.7}=0.
\end{aligned}
$$

Dividing these equations by the respective coefficients of v_0, they become

$$
\begin{aligned}
&v_0+\frac{n}{2}\alpha_0+\frac{n(n-1)}{2.3}\beta_0+\frac{n(n-1)(n-2)}{2.3.4}\gamma_0+\frac{\mathrm{M}}{4(n+1)}=0, \\
&v_0+\frac{2n+1}{3}\alpha_0+\frac{(n-1)(3n+2)}{3.4}\beta_0 \\
&\qquad+\frac{(n-1)(n-2)(4n+3)}{3.4.5}\gamma_0+\frac{\mathrm{P}}{10n(n+1)}=0, \\
&v_0+\frac{3n(n+1)}{2(2n+1)}\alpha_0+\frac{(n-1)(12n^2+15n+2)}{4.5(2n+1)}\beta_0 \\
&\qquad+\frac{(n-1)(n-2)(10n^2+14n+3)}{3.4.5(2n+1)}\gamma_0+\frac{\mathrm{Q}}{10n(n+1)(2n+1)}=0,
\end{aligned}
$$

$$v_0+\frac{2(2n+1)(3n^2+3n-1)}{15n(n+1)}a_0+\frac{(n-1)(10n^3+18n^2+5n-2)}{30n(n+1)}\beta_0$$
$$+\frac{(n-1)(n-2)(20n^3+40n^2+16n-3)}{210n(n+1)}\gamma_0+\frac{R}{35n^2(n+1)^2}=0.$$

Differencing these equations, and then dividing by the respective coefficients of a_0, we get

$$a_0+\frac{n-1}{2}\beta_0+\frac{3(n-1)(n-2)}{20}\gamma_0=\frac{15nM-6P}{10n(n+1)(n+2)}$$
$$a_0+\frac{3n-1}{5}\beta_0+\frac{n(n-2)}{5}\gamma_0=\frac{6(2n+1)P-6Q}{10(n-1)\ldots.(n+2)}$$
$$a_0+\frac{4n^3+3n^2-n-2}{2(3n^2+3n+2)}\beta_0+\frac{(n-2)(10n^3+12n^2+n-3)}{14(3n^2+3n+2)}\gamma_0$$
$$=\frac{21n(n+1)Q-6(2n+1)R}{7(n-1)\ldots.(n+2)(3n^2+3n+2)}.$$

Again differencing, and dividing by the coefficients of β_0,

$$\beta_0+\frac{n-2}{2}\gamma_0=-\frac{15M}{(n+1)(n+2)(n+3)}+\frac{18nP-6Q}{(n-1)\ldots.(n+3)}$$
$$\beta_0+\frac{4n-5}{7}\gamma_0=-\frac{6(3n^2+3n+2)P-12(2n+1)Q}{(n-2)\ldots.(n+3)}-\frac{60R}{7(n-2)\ldots(n+3)}$$
$$\therefore\ \gamma_0=\frac{210M}{(n+1)\ldots.(n+4)}-\frac{84(6n^2-3n+2)P-420nQ+120R}{(n-2)\ldots.(n+4)}$$
$$\beta_0=-\frac{30(4n-5)M}{(n+1)\ldots.(n+4)}+\frac{6(45n^2-9n+14)P-24(9n+1)Q+60R}{(n-1)\ldots.(n+4)}$$
$$a_0=\frac{30(n^2-n+1)M}{(n+1)\ldots.(n+4)}-\frac{6(20n^2+5n+8)P-6(15n+4)Q+24R}{2.n\ldots.(n+4)}$$
$$v_0=-\frac{2(2n+1)(n^2+n+3)M-(6n^2+6n+5)P+2(2n+1)Q-R}{(n+1)\ldots.(n+4)}.$$

Hence, by restoring the foregoing values of M, P, Q, R, we find

$$v_0=\ \Sigma\frac{(x+1)(x+2)(x+3)}{(n+1)(n+2)(n+3)(n+4)}\mu_x$$
$$a_0=-3\Sigma\frac{(x+1)(x+2)\{n+4(x+4)\}}{n(n+1)(n+2)(n+3)(n+4)}\mu_x$$
$$\beta_0=\ 6\Sigma\frac{(x+1)\{n^2+(4x+15)n+(10x^2+66x+104)\}}{(n-1)n\ldots.(n+4)}\mu_x$$
$$\gamma_0=-6\Sigma\frac{n^3+(4x+13)n^2+(10x^2+58x+74)n+(20x^3+160x^2+388x+272)}{(n-2)(n-1)n\ldots.(n+4)}\mu_x.$$

These results are perhaps more systematic when put in the following form :—

$$v_0 = \Sigma\frac{\mu(x+1)(x+2)(x+3)}{(n+1)\ldots.(n+4)}$$

$$\alpha_0 = -3\Sigma\frac{\mu(x+1)(x+2)}{(n+1)\ldots.(n+4)}\left(1+4\frac{x+4}{n}\right)$$

$$\beta_0 = 6\Sigma\frac{\mu(x+1)}{(n+1)\ldots.(n+4)}\left(1+4\frac{x+4}{n}+10\frac{(x+4)(x+3)}{n(n-1)}\right)$$

$$\gamma_0 = -6\Sigma\frac{\mu}{(n+1)\ldots.(n+4)}\left(1+4\frac{x+4}{n}+10\frac{(x+4)(x+3)}{n(n-1)}\right.$$
$$\left.+20\frac{(x+4)(x+3)(x+2)}{n(n-1)(n-2)}\right)\ldots.\ (h).$$

From the values of v_0, α_0, β_0, γ_0, thus determined, the several values of v can be successively deduced by summation according to the scheme of differences, or they may be calculated separately and independently by substitution in the formula (δ).

To thus extend the investigation to the adjustment of higher orders of differences would involve the discussion of equations which increase both in number and complexity, and would evidently be a work of considerable labour. I have, however, by a somewhat novel artifice, succeeded in reducing the fundamental expressions to a form so remarkably unique and comprehensive, that the general solution of the problem, for any order of differences, becomes not merely simplified but divested of all complication whatever.

The primary basis of this improvement consists in prefixing to the scheme a certain set of supplementary differences obtained by a receding process in which the prefixed values of the original function are severally assumed to be zero. These supplementary differences will be placed in the form of a wedge, and will cause the differences of each order to be equal in number to that of the original functions, and impart to them the peculiar property of making the values of the several functions depend exclusively on any stated order of differences according to a formula in which the coefficients are all included in one common factorial expression.

Thus, resuming the case of fourth differences, and retaining an uniformity of notation, the extended scheme will become as follows, where the upper ridge of values v_0, α_{-1}, β_{-2}, γ_{-3}, δd_{-4}, are necessarily equal, though designated by different symbols:—

$$
\begin{array}{c|c|c|c|c}
 & & & & \delta d_{-4} \\
 & & & \gamma_{-3} & \\
 & & \beta_{-2} & & \delta d_{-3} \\
 & \underline{\alpha_{-1}} & & \gamma_{-2} & \\
v_0 & & \underline{\beta_{-1}} & & \delta d_{-2} \\
 & \alpha_0 & & \underline{\gamma_{-1}} & \\
v_1 & & \beta_0 & & \underline{\delta d_{-1}} \\
 & \alpha_1 & & \gamma_0 & \\
v_2 & & \beta_1 & & \delta d_0 \\
 & \alpha_2 & & \gamma_1 & \\
v_3 & & \beta_2 & & \delta d_1 \\
 & \alpha_3 & & \gamma_2 & \\
v_4 & & \beta_3 & & \vdots \\
 & \alpha_4 & & \vdots & \\
v_5 & & \vdots & & \delta d_{n-4} \\
\vdots & \vdots & & \gamma_{n-3} & \\
\vdots & & \beta_{n-2} & & \\
 & \alpha_{n-1} & & & \\
v_n & & & &
\end{array}
$$

Also the formula (δ), when made to commence with δd_{-4}, so as to include the four supplemental differences, will become simply

$$v_m = \Sigma' \frac{(m-\overline{x+3})(m-\overline{x+2})(m-\overline{x+1})}{2.3} \delta d_x \quad \ldots\ldots \quad (\varepsilon).$$

For the coefficient of δd_x being a function of $m-x$, is independent of the epoch; and when the epoch is transferred to v_{-4}, the values of v_0, α_0, β_0, γ_0, are $v_{-4}=0$, $\alpha_{-4}=0$, $\beta_{-4}=0$, $\gamma_{-4}=0$. The terms comprehended under Σ' (accentuated) will now begin with $x=-4$ and end with $x=m-4$.

When Σv_m^2 is a minimum, it has already been shown, equations (e), that

$$\Sigma v_m=0,\ \Sigma m v_m=0,\ \Sigma m^2 v_m=0,\ \Sigma m^3 v_m=0.$$

And as these, when compounded with any finite functions of x, must still vanish, they are evidently equivalent to the following:—

$$\Sigma v_m=0$$
$$\Sigma v_m(m-x)=0$$
$$\Sigma v_m(m-x)(m-\overline{x-1})=0$$
$$\Sigma v_m(m-x)(m-\overline{x-1})(m-\overline{x-2})=0.$$

When the value of v_m, by (ε), is substituted, the several coefficients of δd_x are now of the common factorial form (f); and if, as before, we abbreviate by putting

$$\mu_x = \frac{(n-\overline{x+3})(n-\overline{x+2})(n-\overline{x+1})(n-x)}{2.3} \delta d_x \quad \ldots \quad (g)$$

the summations, after being taken with respect to all consecutive values of m from 0 to n, will give

$$\Sigma'\mu_x=0$$
$$\Sigma'\mu_x(n-\overline{x-1})=0$$
$$\Sigma'\mu_x(n-\overline{x-1})(n-\overline{x-2})=0$$
$$\Sigma'\mu_x(n-\overline{x-1})(n-\overline{x-2})(n-\overline{x-3})=0.$$

And these are again equivalent to

$$\left.\begin{aligned} \Sigma'\mu_x &=0\\ \Sigma' x\mu_x &=0\\ \Sigma' x^2\mu_x &=0\\ \Sigma' x^3\mu_x &=0 \end{aligned}\right\} \;.\;.\;.\;.\;.\;.\;.\;\; (\theta).$$

Hence, when Σ (unaccentuated) does not include values which depend on the four supplementary differences, these last equations will give

$$\begin{aligned} \Sigma\mu_x &= -\mu_{-4}-\mu_{-3}-\mu_{-2}-\mu_{-1}\\ \Sigma x\mu_x &= 4\mu_{-4}+3\mu_{-3}+2\mu_{-2}+\mu_{-1}\\ \Sigma x^2\mu_x &= -16\mu_{-4}-9\mu_{-3}-4\mu_{-2}-\mu_{-1}\\ \Sigma x^3\mu_x &= 64\mu_{-4}+27\mu_{-3}+8\mu_{-2}+\mu_{-1} \end{aligned}$$

for the determination of the four unknown values μ_{-4}, μ_{-3}, μ_{-2}, μ_{-1}, and thence δd_{-4}, δd_{-3}, δd_{-2}, δd_{-1}.

Adding each respective equation to the next following one, we get

$$\begin{aligned} \Sigma(x+1)\mu_x &= 3\mu_{-4}+2\mu_{-3}+\mu_{-2}\\ \Sigma(x+1)x\mu_x &= -12\mu_{-4}-6\mu_{-3}-2\mu_{-2}\\ \Sigma(x+1)x^2\mu_x &= 48\mu_{-4}+18\mu_{-3}+4\mu_{-2}; \end{aligned}$$

and, adding twice each of these to the next following,

$$\begin{aligned} \Sigma(x+1)(x+2)\mu_x &= -6\mu_{-4}-2\mu_{-3}\\ \Sigma(x+1)(x+2)x\mu_x &= 24\mu_{-4}+6\mu_{-3}; \end{aligned}$$

$$\therefore\ \mu_{-4} = \Sigma\frac{(x+1)(x+2)(x+3)}{2.3}\mu_x$$

$$\mu_{-3} = -\Sigma\frac{(x+1)(x+2)(x+4)}{2}\mu_x.$$

Similarly we find

$$\mu_{-2} = \Sigma\frac{(x+1)(x+3)(x+4)}{2}\mu_x$$

$$\mu_{-1} = -\Sigma\frac{(x+2)(x+3)(x+4)}{2.3}\mu_x.$$

But, as assumed in (g),

$$\mu_{-4} = \frac{(n+1)(n+2)(n+3)(n+4)}{2.3}\delta d_{-4}$$

$$\mu_{-3} = \frac{n(n+1)(n+2)(n+3)}{2.3}\delta d_{-3}$$

$$\mu_{-2} = \frac{(n-1)n(n+1)(n+2)}{2.3}\delta d_{-2}$$

$$\mu_{-1} = \frac{(n-2)(n-1)n(n+1)}{2.3}\delta d_{-1};$$

$$\left.\begin{aligned}
\therefore\ \delta d_{-4} &= \Sigma\frac{(x+1)(x+2)(x+3)}{(n+1)(n+2)(n+3)(n+4)}\mu_x \\
\delta d_{-3} &= -3\Sigma\frac{(x+1)(x+2)(x+4)}{n(n+1)(n+2)(n+3)}\mu_x \\
\delta d_{-2} &= 3\Sigma\frac{(x+1)(x+3)(x+4)}{(n-1)n(n+1)(n+2)}\mu_x \\
\delta d_{-1} &= -\Sigma\frac{(x+2)(x+3)(x+4)}{(n-2)(n-1)n(n+1)}\mu_x
\end{aligned}\right\}\ .\ .\ (k),$$

in which the summations under Σ are from $x=0$ to $x=n-4$ inclusive, and the complete form is embodied in each term.

By algebraically differencing back from the symbols v_0, α_0, β_0, γ_0, the algebraic values of the supplementary differences are found as follows:—

$$\begin{array}{c|c|c|c|l}
 & & & & v_0 \qquad\qquad\quad =\delta d_{-4} \\
 & & & v_0 & \\
 & & v_0 & & -3v_0+\alpha_0 \qquad\quad =\delta d_{-3} \\
 & v_0 & & -2v_0+\alpha_0 & \\
v_0 & & -v_0+\alpha_0 & & 3v_0-2\alpha_0+\beta_0 \quad\ =\delta d_{-2} \\
 & \alpha_0 & & v_0-\alpha_0+\beta_0 & \\
 & & \beta_0 & & -\ v_0+\alpha_0-\beta_0+\gamma_0 =\delta d_{-1} \\
 & & & \gamma_0 &
\end{array}$$

$$\left.\begin{aligned}
\therefore\ v_0 &= \delta d_{-4} \\
\alpha_0 &= 3\delta d_{-4}+\delta d_{-3} \\
\beta_0 &= 3\delta d_{-4}+2\delta d_{-3}+\delta d_{-2} \\
\gamma_0 &= \delta d_{-4}+\delta d_{-3}+\delta d_{-2}+\delta d_{-1}
\end{aligned}\right\}\ .\ .\ .\ .\ (l),$$

which, after substituting the values (k), will be found to be equivalent to the formulæ (h), before determined.

The generalization of the foregoing method, for any order (p) of differences, is perfectly analogous to what has been given, and the systematic law which obtains in the form of the final results is such that they may be readily put down for any other order of differences. It will be observed, that in the expressions for $\delta d_{-p}, \ldots \delta d_{-1}$, the numerical coefficients are the respective terms of the expansion of the binomial $(1-1)^{p-1}$; and that in each numerator the absent factor corresponds with the subnumeral indice of δd.

Thus, for sixth differences, we should have

$$v_m=\Sigma'\frac{(m-\overline{x+5})(m-\overline{x+4})(m-\overline{x+3})(m-\overline{x+2})(m-\overline{x+1})}{2.3.4.5}\delta f_x\ .\ .\ (\epsilon''),$$

$$\mu_x=\frac{(n-\overline{x+5})(n-\overline{x+4})(n-\overline{x+3})(n-\overline{x+2})(n-\overline{x+1})(n-x)}{2.3.4.5}\delta f_x\ .\ .\ (g'');$$

and, for the supplemental differences, we should obtain

$$\begin{aligned}
\delta f_{-6} &= \Sigma\frac{(x+1)(x+2)(x+3)(x+4)(x+5)}{(n+1)(n+2)(n+3)(n+4)(n+5)(n+6)}\mu_x \\
\delta f_{-5} &= -5\Sigma\frac{(x+1)(x+2)(x+3)(x+4)(x+6)}{n(n+1)(n+2)(n+3)(n+4)(n+5)}\mu_x \\
\delta f_{-4} &= 10\Sigma\frac{(x+1)(x+2)(x+3)(x+5)(x+6)}{(n-1)n(n+1)(n+2)(n+3)(n+4)}\mu_x \\
\delta f_{-3} &= -10\Sigma\frac{(x+1)(x+2)(x+4)(x+5)(x+6)}{(n-2)(n-1)n(n+1)(n+2)(n+3)}\mu_x \\
\delta f_{-2} &= 5\Sigma\frac{(x+1)(x+3)(x+4)(x+5)(x+6)}{(n-3)(n-2)(n-1)n(n+1)(n+2)}\mu_x \\
\delta f_{-1} &= -\Sigma\frac{(x+2)(x+3)(x+4)(x+5)(x+6)}{(n-4)(n-3)(n-2)(n-1)n(n+1)}\mu_x \quad . \; . \; . \quad (k'').
\end{aligned}$$

PRACTICAL ADAPTATION OF FOURTH DIFFERENCE FORMULÆ.

To simplify the formulæ for calculation, let $x'=(n-4)-x$, and tabulate the factorial functions

$$k=\frac{(x'+1)(x'+2)(x'+3)(x'+4)}{2.3.4},$$

$$p=+\frac{4k_x}{x+4},\quad q=-\frac{12k_x}{x+3},\quad r=+\frac{12k_x}{x+2},\quad s=-\frac{4k_x}{x+1}.$$

Then we shall have

$$\left.\begin{aligned}
\delta d_{-4} &= \Sigma\frac{kp}{k_n}\,\delta d_x \\
\delta d_{-3} &= \Sigma\frac{kq}{k_{n-1}}\,\delta d_x \\
\delta d_{-2} &= \Sigma\frac{kr}{k_{n-2}}\,\delta d_x \\
\delta d_{-1} &= \Sigma\frac{ks}{k_{n-3}}\,\delta d_x
\end{aligned}\right\} \quad . \; . \; . \; . \; . \; (n);$$

also,

$$v_m = p_m\delta d_{-4}+p_{m-1}\delta d_{-3}+p_{m-2}\delta d_{-2} \; . \; . \; . \; +\delta d_{m-4};$$

that is,

$$\begin{aligned}
v_0 &= \delta d_{-4} \\
v_1 &= 4\delta d_{-4}+\delta d_{-3} \\
v_2 &= 10\delta d_{-4}+4\delta d_{-3}+\delta d_{-2} \\
v_3 &= 20\delta d_{-4}+10\delta d_{-3}+4\delta d_{-2}+\delta d_{-1} \\
v_4 &= 35\delta d_{-4}+20\delta d_{-3}+10\delta d_{-2}+4\delta d_{-1}+\delta d_0 \\
&\&\text{c.} \qquad\qquad\qquad \&\text{c.}
\end{aligned}$$

But it will be found that the several values of v may be otherwise readily deduced by successive summation of the columns of differences.

The tables here required are easily prepared, and are as follows:—

Subsidiary Tables for Fourth Differences.

x'	k
&c.	&c.
20	10626
19	8855
18	7315
17	5985
16	4845
15	3876
14	3060
13	2380
12	1820
11	1365
10	1001
9	715
8	495
7	330
6	210
5	126
4	70
3	35
2	15
1	5
0	1

x	p	q	r	s
0	+ 1	− 4	+ 6	− 4
1	4	15	20	10
2	10	36	45	20
3	20	70	84	35
4	35	120	140	56
5	56	189	216	84
6	84	280	315	120
7	120	396	440	165
8	165	540	594	220
9	220	715	780	286
10	286	924	1001	364
11	364	1170	1260	455
12	455	1456	1560	560
13	560	1785	1904	680
14	680	2160	2295	816
15	816	2584	2736	969
16	969	3060	2823	1140
&c.	&c.	&c.	&c.	&c.

These tables may be applied with the greatest facility to the resolution of any given case by means of the following simple rule, suggested by the formulæ (m):—

RULE.—*Beginning with $x'=n$, transcribe the several numbers to the end of Column (k). The first four of these will be the four divisors of the respective formulæ, and the remaining numbers will be the multipliers of each set. Opposite to these multipliers write down as many leading values of p, q, r, s, in four adjoining columns. Then the numerical coefficients of the formulæ expressing the four supplementary fourth differences will be found by taking the horizontal products kp, kq, kr, ks, and dividing the sets by the respective divisors before mentioned.*

Example 1.—Let there be only one fourth difference; then $n=4$, and there will be only one value of x, viz., $x=x'=0$.

The numbers from the tables will stand thus:—

$$k\left\{\begin{array}{r|c|c|c|c|}
70 & \ldots k_n \\
35 \\
15 \\
5 & p & q & r & s \\
\hline
1 & +1 & -4 & +6 & -4
\end{array}\right.$$

$$\therefore\ \delta d_{-4}=+\frac{1}{70}\delta d_0 \qquad \delta d_{-2}=+\frac{6}{15}\delta d_0=+\frac{2}{5}\delta d_0$$

$$\delta d_{-3}=-\frac{4}{35}\delta d_0 \qquad \delta d_{-1}=-\frac{4}{5}\delta d_0.$$

Hence, multiplying by 70 to obviate fractions, the values of v are found by continued summation of each preceding column, as follows; the supplementary differences being cut off by horizontal lines :—

$70\delta d$	70γ	70β	$70a$	$70v$
$+\ \delta d_0$	$+\ \delta d_0$	$+\ \delta d_0$	$+\ \delta d_0$	$+\ \delta d_0$
$-\ 8\delta d_0$	$-\ 7\delta d_0$	$-\ 6\delta d_0$	$-\ 5\delta d_0$	$-4\delta d_0$
$+28\delta d_0$	$+21\delta d_0$	$+15\delta d_0$	$+10\delta d_0$	$+6\delta d_0$
$-56\delta d_0$	$-35\delta d_0$	$-20\delta d_0$	$-10\delta d_0$	$-4\delta d_0$
$+70\delta d_0$	$+35\delta d_0$	$+15\delta d_0$	$+\ 5\delta d_0$	$+\ \delta d_0$

$$\therefore\ v_0=+\frac{1}{70}\delta d_0 \qquad v_2=+\frac{6}{70}\delta d_0 \qquad v_4=\frac{1}{70}\delta d_0$$

$$v_1=-\frac{4}{70}\delta d_0 \qquad v_3=-\frac{4}{70}\delta d_0$$

Example 2.—Let there be two fourth differences, d_0, d_1; then $n=5$, and the tabular numbers are

126				
70				
35				
15	p	q	r	s
5	+1	− 4	+ 6	− 4
1	4	15	20	10

$$\therefore\ \delta d_{-4}=\ \frac{5}{126}\delta d_0+\frac{4}{126}\delta d_1 \qquad \delta d_{-2}=\ \frac{6}{7}\delta d_0+\frac{4}{7}\delta d_1$$

$$\delta d_{-3}=-\frac{4}{14}\delta d_0-\frac{3}{14}\delta d_1 \qquad \delta d_{-1}=-\frac{4}{3}\delta d_0-\frac{2}{3}\delta d_1$$

And, operating only with the numerical coefficients,

$126\delta d$	126γ	126β	$126a$	$126v$
+ 5+ 4	+ 5+ 4	+ 5+ 4	+ 5+ 4	+ 5+ 4
− 36−27	− 31−23	−26−19	−21−15	−16−11
+108+72	+ 77+49	+51+30	+30+15	+14+ 4
−168−84	− 91−35	−40− 5	−10+10	+ 4+14
126 0	+ 35−35	− 5−40	−15−30	−11−16
126	+ 35+91	+30+51	+15+21	+ 4+ 5

$$\therefore\ v_0 = \frac{5}{126}\delta d_0 + \frac{4}{126}\delta d_1 \qquad v_3 = \frac{4}{126}\delta d_0 + \frac{14}{126}\delta d_1$$

$$v_1 = -\frac{16}{126}\delta d_0 - \frac{11}{126}\delta d_1 \qquad v_4 = -\frac{11}{126}\delta d_0 - \frac{16}{126}\delta d_1$$

$$v_2 = \frac{14}{126}\delta d_0 + \frac{4}{126}\delta d_1 \qquad v_5 = \frac{4}{126}\delta d_0 + \frac{5}{126}\delta d_1$$

Example 3.—Let there be three fourth differences; then $n=6$, and

k	p	q	r	s
210† 126 70 35				
15	+ 1	− 4	+ 6	− 4
5	4	15	20	10
1	10	36	45	20

$$\because\ \delta d_{-4} = \Sigma\frac{kp}{210} = \frac{3}{42}\delta d_0 + \frac{4}{42}\delta d_1 + \frac{2}{42}\delta d_2$$

$$\delta d_{-3} = \Sigma\frac{kq}{126} = -\frac{20}{42}\delta d_0 - \frac{25}{42}\delta d_1 - \frac{12}{42}\delta d_2$$

$$\delta d_{-2} = \Sigma\frac{kr}{70} = \frac{18}{14}\delta d_0 + \frac{20}{14}\delta d_1 + \frac{9}{14}\delta d_2$$

$$\delta d_{-1} = \Sigma\frac{ks}{35} = -\frac{12}{7}\delta d_0 - \frac{10}{7}\delta d_1 - \frac{4}{7}\delta d_2$$

And, by summation of the coefficients, as before,

$42\delta d$	42γ	42β	42α	$42v$
+ 3 + 4 + 2	+ 3 + 4 + 2	+ 3 + 4 + 2	+ 3 + 4 + 2	+ 3 + 4 + 2
− 20 − 25 − 12	− 17 − 21 − 10	− 14 − 17 − 8	− 11 − 13 − 6	− 8 − 9 − 4
+ 54 + 60 + 27	+ 37 + 39 + 17	+ 23 + 22 + 9	+ 12 + 9 + 3	+ 4 * − 1
− 72 − 60 − 24	− 35 − 21 − 7	− 12 + 1 + 2	* + 10 + 5	+ 4 + 10 + 4
42 0 0	+ 7 − 21 − 7	− 5 − 20 − 5	− 5 − 10 *	− 1 * + 4
42 0	+ 7 + 21 − 7	+ 2 + 1 − 12	− 3 − 9 − 12	− 4 − 9 − 8
42	+ 7 + 21 + 35	+ 9 + 22 + 23	+ 6 + 13 + 11	+ 2 + 4 + 3

$$\therefore\ v_0 = \frac{3}{42}\delta d_0 + \frac{4}{42}\delta d_1 + \frac{2}{42}\delta d_2$$

$$v_1 = -\frac{8}{42}\delta d_0 - \frac{9}{42}\delta d_1 - \frac{4}{42}\delta d_2$$

---

† The numbers k put on a separate slip may be mechanically placed in position upon the other table, and thus all transcription will be dispensed with.

$$v_2 = \frac{4}{42}\delta d_0 \quad * \quad -\frac{1}{42}\delta d_2$$

$$v_3 = \frac{4}{42}\delta d_0 + \frac{10}{42}\delta d_1 + \frac{4}{42}\delta d_2$$

$$v_4 = -\frac{1}{42}\delta d_0 \quad * \quad +\frac{4}{42}\delta d_2$$

$$v_5 = -\frac{4}{42}\delta d_0 - \frac{9}{42}\delta d_1 - \frac{8}{42}\delta d_2$$

$$v_6 = \frac{2}{42}\delta d_0 + \frac{4}{42}\delta d_1 + \frac{3}{42}\delta d_2$$

The accuracy of the work will in all cases be checked by the symmetry of the coefficients, as well as by the condition $\Sigma v = 0$.

Similarly for four fourth differences, or $n=7$, the results are

$$\delta d_{-4} = \frac{7}{66}\delta d_0 + \frac{12}{66}\delta d_1 + \frac{10}{66}\delta d_2 + \frac{4}{66}\delta d_3$$

$$\delta d_{-3} = -\frac{28}{42}\delta d_0 - \frac{45}{42}\delta d_1 - \frac{36}{42}\delta d_2 - \frac{14}{42}\delta d_3$$

$$\delta d_{-2} = \frac{70}{42}\delta d_0 + \frac{100}{42}\delta d_1 + \frac{75}{42}\delta d_2 + \frac{28}{42}\delta d_3$$

$$\delta d_{-1} = -\frac{28}{14}\delta d_0 - \frac{30}{14}\delta d_1 - \frac{20}{14}\delta d_2 - \frac{7}{14}\delta d_3;$$

and, by summation of the coefficients as before,

$$v_0 = \frac{7}{66}\delta d_0 + \frac{12}{66}\delta d_1 + \frac{10}{66}\delta d_2 + \frac{4}{66}\delta d_3$$

$$v_1 = -\frac{16}{66}\delta d_0 - \frac{159}{7.66}\delta d_1 - \frac{116}{7.66}\delta d_2 - \frac{6}{66}\delta d_3$$

$$v_2 = \frac{4}{66}\delta d_0 - \frac{40}{7.66}\delta d_1 - \frac{59}{7.66}\delta d_2 - \frac{4}{66}\delta d_3$$

$$v_3 = \frac{8}{66}\delta d_0 + \frac{20}{66}\delta d_1 + \frac{80}{7.66}\delta d_2 + \frac{3}{66}\delta d_3$$

$$v_4 = \frac{3}{66}\delta d_0 + \frac{80}{7.66}\delta d_1 + \frac{20}{66}\delta d_2 + \frac{8}{66}\delta d_3$$

$$v_5 = -\frac{4}{66}\delta d_0 - \frac{59}{7.66}\delta d_1 - \frac{40}{7\cdot 66}\delta d_2 + \frac{4}{66}\delta d_3$$

$$v_6 = -\frac{6}{66}\delta d_0 - \frac{116}{7.66}\delta d_1 - \frac{159}{7.66}\delta d_2 - \frac{16}{66}\delta d_3$$

$$v_7 = \frac{4}{66}\delta d_0 + \frac{10}{66}\delta d_1 + \frac{12}{66}\delta d_2 + \frac{7}{66}\delta d_3.$$

Again, for five fourth differences,

$$\delta d_{-4} = \frac{14}{99}\delta d_0 + \frac{28}{99}\delta d_1 + \frac{30}{99}\delta d_2 + \frac{20}{99}\delta d_3 + \frac{7}{99}\delta d_4$$

$$\delta d_{-3} = -\frac{28}{33}\delta d_0 - \frac{35}{22}\delta d_1 - \frac{18}{11}\delta d_2 - \frac{35}{33}\delta d_3 - \frac{4}{11}\delta d_4$$

$$\delta d_{-2} = 2\delta d_0 + \frac{10}{3}\delta d_1 + \frac{45}{14}\delta d_2 + 2\delta d_3 + \frac{2}{3}\delta d_4$$

$$\delta d_{-1} = -\frac{20}{9}\delta d_0 - \frac{25}{9}\delta d_1 - \frac{50}{21}\delta d_2 - \frac{25}{18}\delta d_3 - \frac{4}{9}\delta d_4;$$

$$v_0 = \frac{14}{99}\delta d_0 + \frac{28}{99}\delta d_1 + \frac{30}{99}\delta d_2 + \frac{20}{99}\delta d_3 + \frac{7}{99}\delta d_4$$

$$v_1 = -\frac{28}{99}\delta d_0 - \frac{91}{198}\delta d_1 - \frac{42}{99}\delta d_2 - \frac{25}{99}\delta d_3 - \frac{8}{99}\delta d_4$$

$$v_2 = \frac{2}{99}\delta d_0 - \frac{20}{99}\delta d_1 - \frac{139}{14.33}\delta d_2 - \frac{22}{99}\delta d_3 - \frac{8}{99}\delta d_4$$

$$v_3 = \frac{12}{99}\delta d_0 + \frac{30}{99}\delta d_1 + \frac{40}{7.33}\delta d_2 + \frac{3}{66}\delta d_3 \quad *$$

$$v_4 = \frac{9}{99}\delta d_0 + \frac{30}{99}\delta d_1 + \frac{115}{7.33}\delta d_2 + \frac{30}{99}\delta d_3 + \frac{9}{99}\delta d_4$$

$$v_5 = * \quad + \frac{3}{66}\delta d_1 + \frac{40}{7.33}\delta d_2 + \frac{30}{99}\delta d_3 + \frac{12}{99}\delta d_4$$

$$v_6 = -\frac{8}{99}\delta d_0 - \frac{22}{99}\delta d_1 - \frac{139}{14.33}\delta d_2 - \frac{20}{99}\delta d_3 + \frac{2}{99}\delta d_4$$

$$v_7 = -\frac{8}{99}\delta d_0 - \frac{25}{99}\delta d_1 - \frac{42}{99}\delta d_2 - \frac{91}{198}\delta d_3 - \frac{28}{99}\delta d_4$$

$$v_8 = \frac{7}{99}\delta d_0 + \frac{20}{99}\delta d_1 + \frac{30}{99}\delta d_2 + \frac{28}{99}\delta d_3 + \frac{14}{99}\delta d_4.$$

The formulæ may in like manner be constructed for any proposed number of differences. And, finally, to adjust the given original values V_0, V_1, V_2, &c., the calculated corrections or adjustments v_0, v_1, v_2, &c., are to be respectively applied according to their several algebraic signs.

We have yet to make a further most important improvement and extension of the method, by the introduction of certain considerations, connected with the theory of the differences of algebraic functions, which have the effect of concentrating the general principles of the investigation within the narrowest possible limits, and of finally bringing the subject under a remarkably comprehensive point of view.

As the coefficients of δd_x, in the equations (θ), involve x algebraically to powers not exceeding the seventh, it follows that the eighth difference of any nine successive values of a set of each of these coefficients will be zero. Hence it will appear that these four equations will all of them be fulfilled if the several differences be replaced by any of the following horizontal sets of numerical values:—

δd_{-4}	δd_{-3}	δd_{-2}	$\delta d_{\ 1}$	δd_0	δd_1	δd_2	δd_3	δd_4	δd_5	δd_6	δd_7	
+1	−8	+28	−56	+70	−56	+28	− 8	+ 1	0	0	0	&c.
0	+1	− 8	+28	−56	+70	−56	+28	− 8	+ 1	0	0	&c.
0	0	+ 1	− 8	+28	−56	+70	−56	+28	− 8	+1	0	&c.
0	0	0	+ 1	− 8	+28	−56	+70	−56	+28	−8	+1	&c.
		&c.				&c.			&c.			

for each formula will then, in fact, express the eighth difference of nine consecutive coefficients; and, for the reason just stated, each of these eighth differences of specified algebraic functions must necessarily vanish. Therefore, as the same must hold with any positive or negative multiple of a horizontal set of numbers, it will follow generally that all the requisite conditions (θ) will be satisfied provided that

$$\left.\begin{array}{l}\delta d_{-4}=\phi_0\\ \delta d_{-3}=-8\phi_0+\phi_1\\ \delta d_{-2}=+28\phi_0-8\phi_1+\phi_2\\ \delta d_{-1}=-56\phi_0+28\phi_1-8\phi_2+\phi_3\end{array}\right\}\ \ldots\ (1)$$

$$\left.\begin{array}{l}\delta d_0=+70\phi_0-56\phi_1+28\phi_2-8\phi_3\\ \delta d_1=-56\phi_0+70\phi_1-56\phi_2+28\phi_3\\ \delta d_2=+28\phi_0-56\phi_1+70\phi_2-56\phi_3\\ \delta d_3=-8\phi_0+28\phi_1-56\phi_2+70\phi_3\end{array}\right\}\ \ldots\ (2)$$

$$\left.\begin{array}{l}\delta d_4=\phi_0-8\phi_1+28\phi_2-56\phi_3\\ \delta d_5=\ldots\ldots\ \phi_1-8\phi_2+28\phi_3\\ \delta d_6=\ldots\ldots\ldots\ \phi_2-8\phi_3\\ \delta d_7=\ldots\ldots\ldots\ldots\ \phi_3\end{array}\right\}\ \ldots\ (3)$$

where ϕ_0, ϕ_1, ϕ_2, ϕ_3, represent a set of arbitrary or determinate quantities, which may evidently, in like manner, be extended to any number. These algebraic expressions are moreover the fully developed eighth differences of the series

$$0,\ \phi_0,\ \phi_1,\ \phi_2,\ \phi_3,\ 0;$$

and hence also the original quantities v_0, v_1, v_2, v_3, v_4, &c., are the fourth differences of the same series, and are therefore expressed as follows:—

$$\begin{aligned}
v_0 &= \phi_0 \\
v_1 &= -4\phi_0 + \phi_1 \\
v_2 &= +6\phi_0 - 4\phi_1 + \phi_2 \\
v_3 &= -4\phi_0 + 6\phi_1 - 4\phi_2 + \phi_3 \\
v_4 &= \phi_0 - 4\phi_1 + 6\phi_2 - 4\phi_3 \\
v_5 &= \ldots\ldots \phi_1 - 4\phi_2 + 6\phi_3 \\
v_6 &= \ldots\ldots\ldots\ldots \phi_2 - 4\phi_3 \\
v_7 &= \ldots\ldots\ldots\ldots\ldots\ldots \phi_3 \quad \ldots\ldots (4).
\end{aligned}$$

The general equations, arising out of the principle of least squares, are thus effectually embodied in the simple condition that the required series of corrections (v) shall be the expanded fourth differences of an interior and more concentrated series of values (ϕ), four less in number, or the same in number as that of the fourth differences d_0, d_1, d_2, &c.

This view of the subject may be arrived at in a still more direct and simple manner by a consideration of the initial equations (e) in connexion with the fundamental formula (ϵ).

Suppose four inner sets of differences, denoted by w, y, z, ϕ, to be inserted in the scheme of adjustments, and to be determined from v_0, v_1, v_2, v_3, &c., by successive summation, thus:—

$$\begin{array}{cccccc|c|cccc}
&&&&&&&&&\delta d_{-4} \\
&&&&&&&&\gamma_{-3}& \\
&&&&&&&\beta_{-2}&&\delta d_{-3} \\
&&&&&&\underline{\alpha_{-1}}&&\gamma_{-2}& \\
&&&&&v_0&&\underline{\beta_{-1}}&&\delta d_{-2} \\
&&&&w_0&&\alpha_0&&\underline{\gamma_{-1}}& \\
&&&y_0&&v_1&&\beta_0&&\underline{\delta d_{-1}} \\
&&z_0&&w_1&&\alpha_1&&\gamma_0& \\
&\phi_0&&y_1&&v_2&&\beta_1&&\delta d_0 \\
&&z_1&&w_2&&\alpha_2&&\gamma_1& \\
\text{Foci}&\phi_1&&y_2&&v_3&&\beta_2&&\delta d_1 \\
&\vdots&z_2&&w_3&&\alpha_3&&\gamma_2&\vdots \\
&&\vdots&y_3&&v_4&&\beta_3&\vdots& \\
&\phi_{n-4}&&\vdots&w_4&&\alpha_4&\vdots&&\underline{\delta d_{n-4}} \\
&&z_{n-3}&&\vdots&v_5&\vdots&&\underline{\gamma_{n-3}}& \\
&&&y_{n-2}&&\vdots&&\underline{\beta_{n-2}}&&\delta d_{n-3} \\
&&&&w_{n-1}&&\underline{\alpha_{n-1}}&&\gamma_{n-2}& \\
&&&&&v_n&&\beta_{n-1}&&\delta d_{n-2} \\
&&&&&&\alpha_n&&\gamma_{n-1}& \\
&&&&&&&\beta_n&&\delta d_{n-1} \\
&&&&&&&&\gamma_n& \\
&&&&&&&&&\delta d_n
\end{array}$$

and by formulæ analagous to (ϵ) we shall have

$$\begin{aligned}
w_n &= \Sigma v \\
y_{n-1} &= \Sigma w = \Sigma(n+1-x)v_x \\
z_{n-2} &= \Sigma y = \Sigma \frac{(n+1-x)(n+2-x)}{2} v_x \\
\phi_{n-3} &= \Sigma z = \Sigma \frac{(n+1-x)(n+2-x)(n+3-x)}{2.3} v_x.
\end{aligned}$$

But, by equations (*e*), $\Sigma v = \Sigma xv = \Sigma x^2 v = \Sigma x^3 v = 0$;

$$\therefore w_n = y_{n-1} = z_{n-2} = \phi_{n-3} = 0.$$

Hence these inner sets of differences will severally terminate with the values w_{n-1}, y_{n-2}, z_{n-3}, ϕ_{n-4}, as they appear in the scheme. If, however, any more inner sets of differences were to be constructed, it is evident that, unless $\Sigma x^4 v = 0$, the number of terms, instead of continuing to contract, would become indefinitely prolonged. In consequence of this remarkable peculiarity, which is evidently applicable to any order of differences, the series of values of ϕ are here designated the *focal adjustments* with respect to the differences under consideration.

Whatever be the values of ϕ_0, ϕ_1, ϕ_2, ϕ_3, &c., which form the radical basis of the differencing scheme, Σv^2 will necessarily be the least possible out of all sets of quantities which can have for their fourth differences the series δd_0, δd_1, δd_2, &c., or the more extended series δd_{-4}, δd_{-3}, δd_{-2}, &c. Therefore, as this condition places no restriction upon the values of ϕ_0, ϕ_1, ϕ_2, ϕ_3, &c., they can be made to fulfil any other condition that may be required. The other sets of differences being linear functions of ϕ_0, ϕ_1, ϕ_2, ϕ_3, &c., and exceeding them in number, are, however, not wholly independent, and it is evident that only as many conditions as focal corrections are admissible.

When, for example, the requisite adjustments of four fourth differences are given, the corresponding focal adjustments are determined by the four equations (2), viz.,

$$\left.\begin{array}{r} 70\phi_0 - 56\phi_1 + 28\phi_2 - 8\phi_3 = \delta d_0 \\ -56\phi_0 + 70\phi_1 - 56\phi_2 + 28\phi_3 = \delta d_1 \\ 28\phi_0 - 56\phi_1 + 70\phi_2 - 56\phi_3 = \delta d_2 \\ -8\phi_0 + 28\phi_1 - 56\phi_2 + 70\phi_3 = \delta d_3 \end{array}\right\} \dots\dots (5);$$

which give

$$\left.\begin{array}{l} 462\phi_0 = 49\delta d_0 + 84\delta d_1 + 70\delta d_2 + 28\delta d_3 \\ 462\phi_1 = 84\delta d_0 + 177\delta d_1 + 164\delta d_2 + 70\delta d_3 \\ 462\phi_2 = 70\delta d_0 + 164\delta d_1 + 177\delta d_2 + 84\delta d_3 \\ 462\phi_3 = 28\delta d_0 + 70\delta d_1 + 84\delta d_2 + 49\delta d_3 \end{array}\right\} \dots\dots (p).$$

The preceding and following supplementary differences are likewise found by substitution in (1) and (3), and the results will be found to accord with those of the same example on page 157. Hence also the focal values may be otherwise found by a continuation of the process of summation, with respect to the coefficients, thus :—

$462v$	$462w$	$462y$
49 + 84 + 70 + 28	49 + 84 + 70 + 28	49 + 84 + 70 + 28
− 112 − 159 − 116 − 42	− 63 − 75 − 46 − 14	− 14 + 9 + 24 + 14
28 − 40 − 59 − 28	− 35 − 115 − 105 − 42	− 49 − 106 − 81 − 28
56 + 140 + 80 + 21	21 + 25 − 25 − 21	− 28 − 81 − 106 − 49
21 + 80 + 140 + 56	42 + 105 + 115 + 35	14 + 24 + 9 − 14
− 28 − 59 − 40 + 28	14 + 46 + 75 + 63	+ 28 + 70 + 84 + 49
− 42 − 116 − 159 − 112	− 28 − 70 − 84 − 49	*
28 + 70 + 84 + 49	*	

$462z$	462ϕ
49 + 84 + 70 + 28	49 + 84 + 70 + 28
35 + 93 + 94 + 42	84 + 177 + 164 + 70
− 14 − 13 + 13 + 14	70 + 164 + 177 + 84
− 42 − 94 − 93 − 35	28 + 70 + 84 + 49
− 28 − 70 − 84 − 49	*
*	Focal coefficients.

Here the last set of numbers are the coefficients of the formulæ (p), and the symmetry of the numbers throughout is an effectual test of the accuracy of the work.

The following series of tables, deduced from equations analagous to (5), exhibit the similar coefficients for any number of differences and foci up to seven. As they are all positive, the algebraic sign is not required.

TABLES A.—*Coefficients for deducing Focal Values from their Eighth Differences or from assumed Fourth Difference Adjustments.*

70ϕ

1

126ϕ

5	4
4	5

42ϕ

3	4	2
4	7	4
2	4	3

462ϕ

49	84	70	28
84	177	164	70
70	164	177	84
28	70	84	49

198ϕ

28	56	60	40	14
56	133	156	110	40
60	156	$204\frac{3}{7}$	156	60
40	110	156	133	56
14	40	60	56	28

4290ϕ

756	1680	2100	1800	1050	336
1680	4340	5880	5300	3200	1050
2100	5880	8715	8380	5300	1800
1800	5300	8380	8715	5880	2100
1050	3200	5300	5880	4340	1680
336	1050	1800	2100	1680	756

858ϕ

180	432	600	600	450	240	72
432	1188	1776	1860	1440	786	240
600	1776	2868	3176	2560	1440	450
600	1860	3176	3743	3176	1860	600
450	1440	2560	3176	2868	1776	600
240	786	1440	1860	1776	1188	432
72	240	450	600	600	432	180

The darker lines in each table show how symmetrically certain coefficients distribute themselves within the four sides of each concentric rectangle. Also the several sets of coefficients are precisely the same whether they are read off vertically or horizontally. The symmetrical relations thus exhibited may indeed be inferred from a like symmetry in the coefficients of the original equations from which the tabular numbers are derived; and they present an efficient and useful check on the accuracy of the calculation. An improved method of constructing these tables will be given hereafter. For more than seven values the resulting fractions are so cumbrous that it would be more convenient to tabulate them in decimals.

If the necessary adjustments of a number of fourth differences can be first assigned, the focal adjustments may be calculated with these tabular coefficients according to formulæ similar to (p), and thence the adjustments of the primitive quantities may be found either by successive differencing or by substitution in expressions of the same kind as (4).

It will be observed that the foregoing methods of calculation proceed upon an assumed previous knowledge of the adjustments required to be made in the fourth differences, and that, in rectifying the progression, these must again depend in some measure upon the judgment and discretion of the computer. In order, however, to determine theoretically the values of δd_0, δd_1, δd_2, &c., so as to effect the requisite adjustment of the quantities V with the least possible displacements, we may assume as an available new condition that the sum of the squares of all the amended fourth differences shall also be a minimum. Resuming the case of four foci, which will sufficiently indicate the general principles of the method which follows from this hypothesis, let the differences be carried out more extensively as follows :—

$$
\begin{array}{cccc|c|ccc|c|ccc|c|cccc}
 & & & & & & & & \delta d_{-4} & & & & & & & & \\
 & & & & & & & \gamma_{-3} & & \delta e_{-3} & & & & & & & \\
 & & & & & & \beta_{-2} & & \delta d_{-3} & & \delta f_{-2} & & & & & & \\
 & & & & & \underline{\alpha_{-1}} & & \gamma_{-2} & & \delta e_{-2} & & \underline{\delta g_{-1}} & & & & & \\
 & & & & v_0 & & \underline{\beta_{-1}} & & \delta d_{-2} & & \underline{\delta f_{-1}} & & \delta h_0 & & & & \\
 & & & w_0 & & \alpha_0 & & \underline{\gamma_{-1}} & & \underline{\delta e_{-1}} & & \delta g_0 & & \delta i_0 & & & \\
 & & y_0 & & v_1 & & \beta_0 & & \underline{\delta d_{-1}} & & \delta f_0 & & \delta h_1 & & \delta k_0 & & \\
 & z_0 & & w_1 & & \alpha_1 & & \gamma_0 & & \delta e_0 & & \delta g_1 & & \delta i_1 & & \delta l_0 & \\
\phi_0 & & y_1 & & v_2 & & \beta_1 & & \delta d_0 & & \delta f_1 & & \delta h_2 & & \delta k_1 & & \delta m_0 \\
 & z_1 & & w_2 & & \alpha_2 & & \gamma_1 & & \delta e_1 & & \delta g_2 & & \delta i_2 & & \delta l_1 & \\
\phi_1 & & y_2 & & v_3 & & \beta_2 & & \delta d_1 & & \delta f_2 & & \delta h_3 & & \delta k_2 & & \delta m_1 \\
 & z_2 & & w_3 & & \alpha_3 & & \gamma_2 & & \delta e_2 & & \delta g_3 & & \delta i_3 & & \delta l_2 & \\
\phi_2 & & y_3 & & v_4 & & \beta_3 & & \delta d_2 & & \delta f_3 & & \delta h_4 & & \delta k_3 & & \delta m_2 \\
 & z_3 & & w_4 & & \alpha_4 & & \gamma_3 & & \delta e_3 & & \delta g_4 & & \delta i_4 & & \delta l_3 & \\
\phi_3 & & y_4 & & v_5 & & \beta_4 & & \underline{\delta d_3} & & \delta f_4 & & \delta h_5 & & \delta k_4 & & \delta m_3 \\
 & z_4 & & w_5 & & \alpha_5 & & \underline{\gamma_4} & & \underline{\delta e_4} & & \delta g_5 & & \delta i_5 & & \delta l_4 & \\
 & & y_5 & & v_6 & & \underline{\beta_5} & & \delta d_4 & & \underline{\delta f_5} & & \delta h_6 & & \delta k_5 & & \\
 & & & w_6 & & \underline{\alpha_6} & & \gamma_5 & & \delta e_5 & & \underline{\delta g_6} & & \delta i_6 & & & \\
 & & & & v_7 & & \beta_6 & & \delta d_5 & & \delta f_6 & & \delta h_7 & & & & \\
 & & & & & \alpha_7 & & \gamma_6 & & \delta e_6 & & \delta g_7 & & & & & \\
 & & & & & & \beta_7 & & \delta d_6 & & \delta f_7 & & & & & & \\
 & & & & & & & \gamma_7 & & \delta e_7 & & & & & & & \\
 & & & & & & & & \delta d_7 & & & & & & & &
\end{array}
$$

And let the differences of V_0, V_1, V_2, V_3, &c., be extended in like manner and with an analogous notation, only omitting the symbol δ. Then the complete list of the adjusted fourth differences, expressed in terms of the focal quantities ϕ_0, ϕ_1, ϕ_2, ϕ_3, will be

$$
\begin{array}{l}
d_{-4}+\delta d_{-4}=d_{-4}+\phi_0 \\
d_{-3}+\delta d_{-3}=d_{-3}-8\phi_0+\phi_1 \\
d_{-2}+\delta d_{-2}=d_{-2}+28\phi_0-8\phi_1+\phi_2 \\
d_{-1}+\delta d_{-1}=d_{-1}-56\phi_0+28\phi_1-8\phi_2+\phi_3 \\
\hline
d_0+\delta d_0=d_0+70\phi_0-56\phi_1+28\phi_2-8\phi_3 \\
d_1+\delta d_1=d_1-56\phi_0+70\phi_1-56\phi_2+28\phi_3 \\
d_2+\delta d_2=d_2+28\phi_0-56\phi_1+70\phi_2-56\phi_3 \\
d_3+\delta d_3=d_3-8\phi_0+28\phi_1-56\phi_2+70\phi_3 \\
\hline
d_4+\delta d_4=d_4+\phi_0-8\phi_1+28\phi_2-56\phi_3 \\
d_5+\delta d_5=d_5\,\dots\dots+\phi_1-8\phi_2+28\phi_3 \\
d_6+\delta d_6=d_6\,\dots\dots\dots+\phi_2-8\phi_3 \\
d_7+\delta d_7=d_7\,\dots\dots\dots\dots+\phi_3
\end{array}
$$

The adjustments under consideration are specifically those of the fourth differences d_0, d_1, d_2, d_3; but the four supplementary differences which precede, and also those which follow, are included in this list, in order to avoid any material disturbance of these adjoining differences, by operating upon the results which include the whole of the effects developed by the focal adjustments.

Now, in order to determine the focal quantities ϕ_0, ϕ_1, ϕ_2, &c., when the sum of the squares of these adjusted differences is a minimum, the resolving equations, according to the rule on page 145, will be found by first multiplying each equation by the coefficient of ϕ_0, and equating the sum of the products with zero; and then severally proceeding to operate in the same manner with respect to ϕ_1, ϕ_2, ϕ_3. Hence, observing that

$$d_{-4}-8d_{-3}+28d_{-2}-56d_{-1}+70d_0-56d_1+28d_2-8d_3+d_4$$

is the eighth difference of the series

$$d_{-4},\ d_{-3},\ d_{-2},\ d_{-1},\ d_0,\ d_1,\ d_2,\ d_3,\ d_4,$$

and therefore equal to m_0; and that the other similar sets of products reduce in exactly the same manner, the required equations of condition are simply

$$\left.\begin{array}{l} m_0+\delta m_0=0 \\ m_1+\delta m_1=0 \\ m_2+\delta m_2=0 \\ m_3+\delta m_3=0 \end{array}\right\} \text{or} \left\{\begin{array}{l} \delta m_0=-m_0 \\ \delta m_1=-m_1 \\ \delta m_2=-m_2 \\ \delta m_3=-m_3 \end{array}\right\} \quad . \ . \ . \ (w)$$

These show that the twelfth differences of V_0, V_1, V_2, &c., must be neutralized by the sixteenth differences of ϕ_0, ϕ_1, ϕ_2, &c.

The sixteenth differences, viz., δm_0, δm_1, δm_2, δm_3, being thus known, the values of ϕ_0, ϕ_1, ϕ_2, ϕ_3, are, according to the table of coefficients, page 311, vol. xi., determined by the equations

$$\left.\begin{array}{r} 12870\phi_0-11440\phi_1+\ 8008\phi_2-\ 4368\phi_3=\delta m_0 \\ -11440\phi_0+12870\phi_1-11440\phi_2+\ 8008\phi_3=\delta m_1 \\ 8008\phi_0-11440\phi_1+12870\phi_2-11440\phi_3=\delta m_2 \\ -4368\phi_0+\ 8008\phi_1-11440\phi_2+12870\phi_3=\delta m_3 \end{array}\right\} \ . \ . \ (6)$$

By solving these equations we get the following formulæ for computation with four foci:—

$$\left.\begin{array}{l} 277134\phi_0=\ 605\delta m_0+1320\delta m_1+1188\delta m_2+\ 440\delta m_3 \\ 277134\phi_1=1320\delta m_0+3165\delta m_1+3048\delta m_2+1188\delta m_3 \\ 277134\phi_2=1188\delta m_0+3048\delta m_1+3165\delta m_2+1320\delta m_3 \\ 277134\phi_3=\ 440\delta m_0+1188\delta m_1+1320\delta m_2+\ 605\delta m_3 \end{array}\right\} \ . \ . \ (9)$$

The determination of these coefficients is indeed evidently the same as would be obtained by the former method when applied to eighth instead of fourth differences; and they are here similarly tabulated up to seven differences, or seven foci.

TABLES B.—*Coefficients for deducing Focal Adjustments from Twelfth Differences.*

12870.ϕ

1

24310.ϕ

9	8
8	9

24310.ϕ

25	40	20
40	73	40
20	40	25

277134.ϕ

605	1320	1188	440
1320	3165	3048	1188
1188	3048	3165	1320
440	1188	1320	605

$8398.\phi$

33	88	108	72	22
88	253	328	228	72
108	328	$449\frac{4}{11}$	328	108
72	228	328	253	88
22	72	108	88	33

$293930.\phi$

1859	5720	8580	7800	4290	1144
5720	18755	29480	27780	15720	4290
8580	29480	48455	47480	27780	7800
7800	27780	47480	48455	29480	8580
4290	15720	27780	29480	18755	5720
1144	4290	7800	8580	5720	1859

$125970.\phi$

1183	4056	7020	7800	5850	2808	676
4056	14703	26520	30420	23400	11466	2808
7020	26520	49695	58920	46620	23400	5850
7800	30420	58920	72195	58920	30420	7800
5850	23400	46620	58920	49695	26520	7020
2808	11466	23400	30420	26520	14703	4056
676	2808	2850	7800	7020	4056	1183

The coefficients of each higher table may, with a considerable saving of calculation, be deduced from those of the table immediately preceding it by the following method:—

Distinguishing by accentuation the symbols which appertain to the preceding table, let the equations analogous to (6) be

$$\left.\begin{array}{l}
A_0\phi'_0 - A_1\phi'_1 + A_2\phi'_2 \ldots . \mp A_{n-1}\phi'_{n-1} = \delta m_0 \\
-A_1\phi'_0 + A_0\phi'_1 - A_1\phi'_2 \ldots . \pm A_{n-2}\phi'_{n-1} = \delta m_1 \\
A_2\phi'_0 - A_1\phi'_1 + A_0\phi'_2 \ldots . \mp A_{n-3}\phi'_{n-1} = \delta m_2 \\
\vdots \qquad \vdots \qquad \vdots \qquad \qquad \vdots \qquad \vdots \\
\mp A_{n-1}\phi'_0 \pm A_{n-2}\phi'_1 \mp A_{n-3}\phi'_2 \ldots + A_0\phi'_{n-1} = \delta m_{n-1}
\end{array}\right\} \ldots (7)$$

where A_0, A_1, A_2, &c., are factorial functions of n; and let the solution of these equations, as given by the tabular coefficients, be denoted thus:—

$$\left.\begin{array}{l} \phi'_0 = \phi'_{0,0}\delta m_0 + \phi'_{0,1}\delta m_1 + \phi'_{0,2}\delta m_2 \dots + \phi'_{0,n-1}\delta m_{n-1} \\ \phi'_1 = \phi'_{1,0}\delta m_0 + \phi'_{1,1}\delta m_1 + \phi'_{1,2}\delta m_2 \dots + \phi'_{1,n-1}\delta m_{n-1} \\ \phi'_2 = \phi'_{2,0}\delta m_0 + \phi'_{2,1}\delta m_1 + \phi'_{2,2}\delta m_2 \dots + \phi'_{2,n-1}\delta m_{n-1} \\ \vdots \\ \phi'_{n-1} = \phi'_{n-1\ 0}\delta m_0 + \phi'_{n-1,1}\delta m_1 + \phi'_{n-1,2}\delta m_2 \dots + \phi'_{n-1,n-1}\delta m_{n-1} \end{array}\right\} \dots (8)$$

Let the equations of condition which have to determine the coefficients of the sought table be

$$\left.\begin{array}{l} A_0\phi_0 - A_1\phi_1 + A_2\phi_2 \dots \mp A_{n-1}\phi_{n-1} \pm A_n\phi_n = \delta m_0 \\ -A_1\phi_0 + A_0\phi_1 - A_1\phi_2 \dots \pm A_{n-2}\phi_{n-1} \mp A_{n-1}\phi_n = \delta m_1 \\ A_2\phi_0 - A_1\phi_1 + A_0\phi_2 \dots \mp A_{n-3}\phi_{n-1} \pm A_{n-2}\phi_n = \delta m_2 \\ \mp A_{n-1}\phi_0 \pm A_{n-2}\phi_1 \mp A_{n-3}\phi_2 \dots + A_0\phi_{n-1} - A_1\phi_n = \delta m_{n-1} \\ \pm A_n\phi_0 \mp A_{n-1}\phi_1 \pm A_{n-2}\phi_2 \dots - A_1\phi_{n-1} + A_0\phi_n = \delta m_n \end{array}\right\} \dots (9)$$

Now, if $\delta m_0 \mp A_n\phi_n$, $\delta m_1 \pm A_{n-1}\phi_n$, $\delta m_2 \mp A_{n-2}\phi_n$, ... $\delta m_{n-1} + A_1\phi_n$ be substituted in place of δm_0, δm_1, δm_2, ... δm_{n-1} respectively in the equation (7), they will be identical in form with the equations (9), omitting for the present the last one. Hence, by a like substitution in the first of the formulæ (8), we get

$$\begin{aligned} \phi_0 &= \phi'_{0,0}(\delta m_0 \mp A_n\phi_n) + \phi'_{0,1}(\delta m_1 \pm A_{n-1}\phi_n) \\ &\quad + \phi'_{0,2}(\delta m_2 \mp A_{n-2}\phi_n) \dots + \phi'_{0,n-1}(\delta m_{n-1} + A_1\phi_n) \\ &= \phi'_0 + (A_1\phi'_{0,n-1} - A_2\phi'_{0,n-2} + A_3\phi'_{0,n-3} - \&c.)\phi_n. \end{aligned}$$

And in this way the formulæ (8) severally give

$$\left.\begin{array}{l} \phi_0 = \phi'_0 + \lambda_0\phi_n \\ \phi_1 = \phi'_1 + \lambda_1\phi_n \\ \phi_2 = \phi'_2 + \lambda_2\phi_n \\ \vdots \\ \phi_{n-1} = \phi'_{n-1} + \lambda_{n-1}\phi_n \end{array}\right\} \dots\dots (10)$$

where

$$\left.\begin{array}{l} \lambda_0 = A_1\phi'_{0,n-1} - A_2\phi'_{0,n-2} + A_3\phi'_{0,n-3} - \&c. \\ \lambda_1 = A_1\phi'_{1,n-1} - A_2\phi'_{1,n-2} + A_3\phi'_{1,n-3} - \&c. \\ \lambda_2 = A_1\phi'_{2,n-1} - A_2\phi'_{2,n-2} + A_3\phi'_{2,n-3} - \&c. \\ \vdots \\ \lambda_{n-1} = A_1\phi'_{n-1,n-1} - A_2\phi'_{n-1,n-2} + A_3\phi'_{n-1,n-3} - \&c. \\ \quad = A_1\phi'_{0,0} - A_2'\phi'_{0,1} + A_3\phi'_{0,2} - \&c. \end{array}\right\} \dots (11)$$

The last of the equations (9) now gives

$$\pm \mathrm{A}_n(\phi'_0+\lambda_0\phi_n)\mp \mathrm{A}_{n-1}(\phi'_1+\lambda_1\phi_n)\pm \mathrm{A}_{n-2}(\phi'_2+\lambda_2\phi_n)$$
$$\ldots\ldots -\mathrm{A}_1(\phi'_{n-1}+\lambda_{n-1}\phi_n)+\mathrm{A}_0\phi_n=\delta m_n.$$

But

$$\pm \mathrm{A}_n\phi'_0\mp \mathrm{A}_{n-1}\phi'_1\pm \mathrm{A}_{n-2}\phi'_2 \ldots\ldots -\mathrm{A}_1\phi'_{n-1}$$
$$=-\mathrm{A}_1(\phi'_{n-1,\,0}\delta m_0+\phi'_{n-1,\,1}\delta m_1+\phi'_{n-1,\,2}\delta m_2+\&\mathrm{c.})$$
$$+\mathrm{A}_2(\phi'_{n-2,\,0}\delta m_0+\phi'_{n-2,\,1}\delta m_1+\phi'_{n-2,\,2}\delta m_2+\&\mathrm{c.})$$
$$-\mathrm{A}_3(\phi'_{n-3,\,0}\delta m_0+\phi'_{n-3,\,1}\delta m_1+\phi'_{n-3,\,2}\delta m_2+\&\mathrm{c.})$$
$$\&\mathrm{c.} \qquad\qquad \&\mathrm{c.}$$
$$=-(\lambda_0\delta m_0+\lambda_1\delta m_1+\lambda_2\delta m_2 \ldots\ldots +\lambda_{n-1}\delta m_{n-1});$$
$$\therefore\ (\mathrm{A}_0-\mathrm{A}_1\lambda_{n-1}+\mathrm{A}_2\lambda_{n-2}\ .\ ,\ .\ .\ \pm \mathrm{A}_n\lambda_0)\phi_n$$
$$=\lambda_0\delta m_0+\lambda_1\delta m_1+\lambda_2\delta m_2 \ldots\ldots +\lambda_{n-1}\delta m_{n-1}+\delta m_n.$$

Hence, as the coefficient of δm_n in the value of ϕ_n is $\phi_{n,\,n}=\phi_{0,\,0}$, we must have

$$\phi_n=\phi_{0,\,0}(\lambda_0\delta m_0+\lambda_1\delta m_1+\lambda_2\delta m_2 \ldots\ldots +\lambda_{n-1}\delta m_{n-1}+\delta m_n)\ \ .\ \ .\ \ (12)$$

where the several coefficients are $\lambda_0\phi_{0,\,0}$, $\lambda_1\phi_{0,\,0}$, $\lambda_2\phi_{0,\,0}$, . . . $\phi_{0,\,0}$, and these must be the same as the coefficients of ϕ_0, taken in a reverse order. But ϕ_0 is the same as the first supplementary difference in the scheme, for which a general formula has before been found.

If p denote the order of the differencing, counted from ϕ to the differences to be adjusted, then, according to (k), n being now $n+p$,

$$\phi_0=\Sigma\frac{x+1\ \ldots\ x+p-1}{n+p+1\ldots n+2p}\cdot\frac{n+1-x\ \ldots\ n+p-x}{2\ \ldots\ p-1}\,\delta m_x\ .\ \ .\ \ (13)$$

As the coefficients of ϕ_n are those of ϕ_0 reversed, put $n-x$ for x, and

$$\phi_n=\Sigma\frac{n-x+1\ \ldots\ n-x+p-1}{n+p+1\ \ldots\ n+2p}\cdot\frac{x+1\ \ldots\ x+p}{2\ \ldots\ p-1}\,\delta m_x\ .\ .\ (14);$$

the respective coefficients of which are

$$\phi_{0,\,n}\quad=\lambda_0\phi_{0,\,0}=p\frac{n+1\ \ldots\ n+p-1}{n+p+1\ \ldots\ n+2p}$$

$$\phi_{0,\,n-1}=\lambda_1\phi_{0,\,0}=p\frac{p+1}{1}\cdot\frac{n\ \ldots\ n+p-2}{n+p+1\ \ldots\ n+2p}$$

$$\phi_{0,\,n-2}=\lambda_2\phi_{0,\,0}=p\frac{p+1}{1}\cdot\frac{p+2}{2}\cdot\frac{n-1\ \ldots\ n+p-3}{n+p+1\ \ldots\ n+2p}$$

$$\phi_{0,\,n-3}=\lambda_3\phi_{0,\,0}=p\frac{p+1}{1}\cdot\frac{p+2}{2}\cdot\frac{p+3}{3}\cdot\frac{n-2\ \ldots\ n+p-4}{n+p+1\ \ldots\ n+2p}$$

$$\vdots \qquad\qquad \vdots \qquad\qquad\qquad\qquad \vdots$$

$$\phi_{0,\,3}\quad=\lambda_{n-3}\phi_{0,\,0}=\frac{p}{1}\cdot\frac{p+1}{2}\cdot\frac{p+2}{3}\cdot\frac{n-2\ \ldots\ n+p-3}{n+p+1\ \ldots\ n+2p}$$

$$\phi_{0,2}=\lambda_{n-2}\phi_{0,0}=\frac{p}{1}\cdot\frac{p+1}{2}\cdot\frac{n-1\ldots n+p-2}{n+p+1\ldots n+2p}$$

$$\phi_{0,1}=\lambda_{n-1}\phi_{0,0}=\frac{p}{1}\cdot\frac{n\ldots n+p-1}{n+p+1\ldots n+2p}$$

$$\phi_{0,0}=\frac{n+1\ldots n+p}{n+p+1\ldots n+2p}=\frac{(n+p)^2}{n(n+2p)}\phi'_{0,0}\ldots(15)$$

$$\therefore\lambda_0=\frac{p}{n+p}$$

$$\lambda_1=p\frac{p+1}{1}\cdot\frac{n}{(n+p)(n+p-1)}$$

$$\lambda_2=p\frac{p+1}{1}\cdot\frac{p+2}{2}\cdot\frac{n(n-1)}{(n+p)(n+p-1)(n+p-2)}$$

$$\lambda_3=p\frac{p+1}{1}\cdot\frac{p+2}{2}\cdot\frac{p+3}{3}\cdot\frac{n(n-1)(n-2)}{(n+p)(n+p-1)(n+p-2)(n+p-3)}$$

$$\begin{array}{ll}\vdots & \qquad\qquad\vdots\end{array}$$

$$\lambda_{n-3}=\frac{p}{1}\cdot\frac{p+1}{2}\cdot\frac{p+2}{3}\cdot\frac{n(n-1)(n-2)}{(n+p)(n+p-1)(n+p-2)}$$

$$\lambda_{n-2}=\frac{p}{1}\cdot\frac{p+1}{2}\cdot\frac{n(n-1)}{(n+p)(n+p-1)}$$

$$\lambda_{n-1}=\frac{p}{1}\cdot\frac{n}{n+p}$$

$$\lambda_n=1\ \ldots\ldots\ldots\ (16)$$

These values may be conveniently made to depend successively upon each other, as follows:—

$$\begin{array}{ll}
\lambda_n=1 & \lambda_0=\dfrac{p}{n+p}\lambda_n \\[2ex]
\lambda_{n-1}=\dfrac{n}{1}\lambda_0 & \lambda_1=\dfrac{p+1}{n+p-1}\lambda_{n-1} \\[2ex]
\lambda_{n-2}=\dfrac{n-1}{2}\lambda_1 & \lambda_2=\dfrac{p+2}{n+p-2}\lambda_{n-2} \\[2ex]
\lambda_{n-3}=\dfrac{n-2}{3}\lambda_2 & \lambda_3=\dfrac{p+3}{n+p-3}\lambda_{n-3} \\[2ex]
\text{\&c.} & \text{\&c.}\quad\ldots\ldots(17)
\end{array}$$

The coefficients are hence all easily deduced by means of the formulæ (12) and (10).

Moreover, according to the formula (10) the absolute value of the function ϕ_x is obviously $\Sigma(\lambda_x\phi_n)$, in which the terms under Σ are determined by giving to n the descending series of values n, $n-1$, $n-2$, &c. Hence from the foregoing general expressions for ϕ_n and the factors λ, we are thus enabled to obtain, for the independent and direct calculation of any focal coefficient, the following general formula:—

$$\phi_{x,y}=\frac{n-x+1\ldots n-x+p}{2\ldots p-1}\cdot\frac{n-y+1\ldots n-y+p}{2\ldots p-1}\times$$

$$\Sigma\frac{(x+1\ldots x+p-1)(y+1\ldots y+p-1)}{n+1\ldots n+2p}\quad\ldots\ (18)$$

where the terms under Σ are to be found by successively diminishing by unity every factor in both numerator and denominator. The number of these terms after the first will evidently be the lesser of the two values x, y. Also if $x'=n-x$, $y'=n-y$, be complementary numbers of the table, since then $\phi_{x,y}=\phi_{x',y'}$, the calculation is simplified, and consists of less terms, if we choose from these two equivalent forms that which contains the least of the four values x, y, x', y'.

The formula (18) expresses the value of the focal coefficient which belongs to a table constructed for $n+1$ differences. Let this, for the present, be distinguished by placing the symbol n as an ordinal exponent, and not as a power; thus $\phi^n_{x,y}$ designates the $y+1$th coefficient of the $x+1$th line of a table for $n+1$ values. Then, by applying the formula to express the several values of the following coefficients, viz.,

$$\phi^n_{x,y},\ \phi^{n-1}_{x-1,y-1},\ \phi^{n-2}_{x-2,y-2},\ \ldots\ \phi^{n-y}_{x-y,0},\ 0,$$

the leading factors, which precede Σ, will remain unaltered, but each succeeding series under Σ will become divested of its first term. The successive terms of the formula, taken separately, must, therefore, be the respective values of the first differences or decrements of the stated series of functions. Hence, by dividing the first of these terms by the second, we find that

$$\phi^n_{x,y}-\phi^{n-1}_{x-1,y-1}=\frac{n}{n+2p}\cdot\frac{x+p-1}{x}\cdot\frac{y+p-1}{y}(\phi^{n-1}_{x-1,y-1}-\phi^{n-2}_{x-2,y-2})$$

$$\therefore\ \phi^n_{x,y}=\phi^{n-1}_{x-1,y-1}+\frac{n}{n+2p}\cdot\frac{x+p-1}{x}\cdot\frac{y+p-1}{y}(\phi^{n-1}_{x-1,y-1}-\phi^{n-2}_{x-2,y-2})\,..\,(19)$$

which supplies an easy method of calculating a coefficient for any table from those contained in the two tables immediately preceding it.

In extreme and exceptional cases, we have

$$\left.\begin{aligned}\phi^n_{x,1}&=\left(1+\frac{n}{n+2p}\cdot\frac{x+p-1}{x}\cdot\frac{p}{1}\right)\phi^{n-1}_{x-1,0}\\ \phi^n_{x,0}&=\frac{n+p}{n+2p}\cdot\frac{x+p-1}{x}\cdot\phi^{n-1}_{x-1,0}\\ \phi^n_{0,0}&=\frac{n+p}{n+2p}\cdot\frac{n+p}{n}\,\phi^{n-1}_{0,0}\end{aligned}\right\}\ \ldots\ (20)$$

For tables A, $p=4$; for tables B, $p=8$.

Examples.—In the table B for seven values, we have

$$\phi^6_{4,2}=\phi^5_{3,1}+\frac{6}{22}\cdot\frac{11}{4}\cdot\frac{9}{2}(\phi^5_{3,1}-\phi^4_{2,0})$$

$$=\frac{2778}{29393}+\frac{27}{8}\left(\frac{2778}{29393}-\frac{54}{4199}\right)$$

$$=\frac{10878}{29393}=\frac{1554}{4199}\left(=\frac{46620}{125970}\right);$$

$$\phi^6_{4,1}=\left(1+\frac{6}{22}\cdot\frac{11}{4}\cdot\frac{8}{1}\right)\phi^5_{3,0}$$

$$=7\frac{780}{29393}=\frac{780}{4199}\left(=\frac{23400}{125970}\right);$$

$$\phi^6_{4,0}=\frac{14}{22}\cdot\frac{11}{4}\cdot\phi^5_{3,0}$$

$$=\frac{7}{4}\cdot\frac{780}{29393}=\frac{195}{4199}\left(=\frac{5850}{125970}\right).$$

Also in relation to the tables B, the computed values of $\phi_{0,0}$, λ_0, λ_1, &c., for a few values of n are here appended:—

n	$\phi_{0,0}$	λ_0	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8	λ_9	λ_{10}
1	$\frac{9}{17.1430}$	$\frac{8}{9}$	1									
2	$\frac{5}{17.286}$	$\frac{4}{5}$	$\frac{8}{5}$	1								
3	$\frac{55}{17.19.78}$	$\frac{8}{11}$	$\frac{108}{55}$	$\frac{24}{11}$	1							
4	$\frac{33}{17.19.26}$	$\frac{2}{3}$	$\frac{24}{11}$	$\frac{36}{11}$	$\frac{8}{3}$	1						
5	$\frac{143}{17.19.70}$	$\frac{8}{13}$	$\frac{30}{13}$	$\frac{600}{143}$	$\frac{60}{13}$	$\frac{40}{13}$	1					
6	$\frac{91}{17.19.30}$	$\frac{4}{7}$	$\frac{216}{91}$	$\frac{450}{91}$	$\frac{600}{91}$	$\frac{540}{91}$	$\frac{24}{7}$	1				
7	$\frac{195}{17.19.46}$	$\frac{8}{15}$	$\frac{12}{5}$	$\frac{72}{13}$	$\frac{110}{13}$	$\frac{120}{13}$	$\frac{36}{5}$	$\frac{56}{15}$	1			
8	$\frac{130}{17.19.23}$	$\frac{1}{2}$	$\frac{12}{5}$	6	$\frac{132}{13}$	$\frac{165}{13}$	12	$\frac{42}{5}$	4	1		
9	$\frac{17.26}{19.23.45}$	$\frac{8}{17}$	$\frac{81}{34}$	$\frac{108}{17}$	$\frac{198}{17}$	$\frac{12.297}{13.17}$	$\frac{297}{17}$	$\frac{252}{17}$	$\frac{162}{17}$	$\frac{72}{7}$	1	
10	$\frac{17.18}{19.23.25}$	$\frac{4}{9}$	$\frac{40}{17}$	$\frac{225}{24}$	$\frac{220}{17}$	$\frac{330}{17}$	$\frac{396}{17}$	$\frac{385}{17}$	$\frac{300}{17}$	$\frac{180}{17}$	$\frac{40}{9}$	1

We shall conclude with a summary recapitulation of the general principles of calculation, according to the method propounded in the last investigation.

Let the given series of numbers, about the vicinity of any objectionable irregularities amongst the fourth differences, be differenced partially out to the twelfth order of differences. For example, in the following scheme, as in that on page 164, four refractory fourth differences (d_0, d_1, d_2, d_3) are considered to be centres of a disturbance which demands adjustment.

$$\begin{array}{c|ccc|c|ccc|c|cccc}
 & & & & d_{-4} & & & & & & & & \\
 & & & c_{-3} & & e_{-3} & & & & & & & \\
 & & b_{-2} & & d_{-3} & & f_{-2} & & & & & & \\
 & a_{-1} & & c_{-2} & & e_{-2} & & g_{-1} & & & & & \\
V_0 & & b_{-1} & & d_{-2} & & f_{-1} & & h_0 & & & & \\
 & a_0 & & c_{-1} & & e_{-1} & & g_0 & & i_0 & & & \\
V_1 & & b_0 & & d_{-1} & & f_0 & & h_1 & \cdot & k_0 & & \\
 & a_1 & & c_0 & & e_0 & & g_1 & & i_1 & & l_0 & \\
V_2 & & b_1 & & d_0 & & f_1 & & h_2 & \cdot & k_1 & & m_0 \\
 & a_2 & & c_1 & & e_1 & & g_2 & & i_2 & & l_1 & \\
V_3 & & b_2 & & d_1 & & f_2 & & h_3 & \cdot & k_2 & & m_1 \\
 & a_3 & & c_2 & & e_2 & & g_3 & & i_3 & & l_2 & \\
V_4 & & b_3 & & d_2 & & f_3 & & h_4 & \cdot & k_3 & & m_2 \\
 & a_4 & & c_3 & & e_3 & & g_4 & & i_4 & & l_3 & \\
V_5 & & b_4 & & d_3 & & f_4 & & h_5 & \cdot & k_4 & & m_3 \\
 & a_5 & & c_4 & & e_4 & & g_5 & & i_5 & & l_4 & \\
V_6 & & b_5 & & d_4 & & f_5 & & h_6 & \cdot & k_5 & & \\
 & a_6 & & c_5 & & e_5 & & g_6 & & i_6 & & & \\
V_7 & & b_6 & & d_5 & & f_6 & & h_7 & & & & \\
 & a_7 & & c_6 & & e_6 & & g_7 & & & & & \\
 & & b_7 & & d_6 & & f_7 & & & & & & \\
 & & & c_7 & & e_7 & & & & & & & \\
 & & & & d_7 & & & & & & & &
\end{array}$$

Thus, by carrying out the differencing, the four twelfth differences m_0, m_1, m_2, m_3 are found; and after reversing their algebraic signs, according to equations (w), they will then become the values of δm_0, δm_1, δm_2, δm_3.

Now, by table B for four differences, the focal values are

$$\phi_0=\frac{605\delta m_0+1320\delta m_1+1188\delta m_2+440\delta m_3}{277134}$$

$$\phi_1=\frac{1320\delta m_0+3165\delta m_1+3048\delta m_2+1188\delta m_3}{277134}$$

$$\phi_2=\frac{1188\delta m_0+3048\delta m_1+3165\delta m_2+1320\delta m_3}{277134}$$

$$\phi_3=\frac{440\delta m_0+1188\delta m_1+1320\delta m_2+605\delta m_3}{277134}.$$

Having calculated these numerically, their expanded fourth differences are then to be developed as in the scheme on page 164, and thus the sought values $(v_0, \ldots v_7)$ of the corrections or adjustments of eight primitive numbers $(V_0, \ldots V_7)$ are determined.

These adjustments will affect the values of twelve fourth differences $(d_{-4}, \ldots d_7)$, which thereby become so modified and reduced in magnitude that the sum of their squares is less than would have similarly resulted from any other values of the focal quantities $(\phi_0, \phi_1, \phi_2, \phi_3)$; and at the same time the values of the principal adjustments $(v_0, \ldots v_7)$ are such that the sum of their squares is less than that of any other set of numbers that can produce, as fourth differences, the four middle adjustments $(\delta d_0, \delta d_1, \delta d_2, \delta d_3)$.

If, according to the same method, second differences be used, instead of fourth, the operations will be perfectly analogous, but will, of course, involve less calculation. In the corresponding case of four centres, the schemes will be as follows :—

$$\begin{array}{|c|c|c|c|c|c c}
 & & b_{-2} & & & & \\
 & \underline{a_{-1}} & & \underline{c_{-1}} & & & \\
V_0 & & \underline{b_{-1}} & & d_0 & & \\
 & a_0 & & c_0 & & e_0 & \\
V_1 & & b_0 & & d_1 & & f_0 \\
 & a_1 & & c_1 & & e_1 & \\
V_2 & & b_1 & & d_2 & & f_1 \\
 & a_2 & & c_2 & & e_2 & \\
V_3 & & b_2 & & d_3 & & f_2 \\
 & a_3 & & c_3 & & e_3 & \\
V_4 & & b_3 & & d_4 & & f_3 \\
 & a_4 & & c_4 & & e_4 & \\
V_5 & & \overline{b_4} & & d_5 & & \\
 & \overline{a_5} & & \overline{c_5} & & & \\
 & & b_5 & & & &
\end{array}
\qquad
\begin{array}{l}
\delta f_0 = -f_0 \\
\delta f_1 = -f_1 \\
\delta f_2 = -f_2 \\
\delta f_3 = -f_3
\end{array}$$

$$\begin{array}{c c|c|c|c|c|c|c c}
 & & & & \delta b_{-2} & & & & \\
 & & & \underline{a_{-1}} & & \underline{\delta c_{-1}} & & & \\
 & & v_0 & & \underline{\delta b_{-1}} & & \delta d_0 & & \\
 & w_0 & & a_0 & & \delta c_0 & & \delta e_0 & \\
\phi_0 & & v_1 & & \delta b_0 & & \delta d_1 & & \delta f_0 \\
 & w_1 & & a_1 & & \delta c_1 & & \delta e_1 & \\
\phi_1 & & v_2 & & \delta b_1 & & \delta d_2 & & \delta f_1 \\
 & w_2 & & a_2 & & \delta c_2 & & \delta e_2 & \\
\phi_2 & & v_3 & & \delta b_2 & & \delta d_3 & & \delta f_2 \\
 & w_3 & & a_3 & & \delta c_3 & & \delta e_3 & \\
\phi_3 & & v_4 & & \delta b_3 & & \delta d_4 & & \delta f_3 \\
 & w_4 & & a_4 & & \delta c_4 & & \delta e_4 & \\
 & & v_5 & & \overline{\delta b_4} & & \delta d_5 & & \\
 & & & \overline{a_5} & & \overline{\delta c_5} & & & \\
 & & & & \delta b_5 & & & &
\end{array}$$

and the formulæ for finding the focal values from the reversed sixth differences will be, according to the table A for four values, viz.:—

$$\phi_0 = \frac{49\delta f_0 + 84\delta f_1 + 70\delta f_2 + 28\delta f_3}{462}$$

$$= \frac{7\delta f_0 + 12\delta f_1 + 10\delta f_2 + 4\delta f_3}{66},$$

$$\phi_1 = \frac{84\delta f_0 + 177\delta f_1 + 164\delta f_2 + 70\delta f_3}{462},$$

$$\phi_2=\frac{70\delta f_0+164\delta f_1+177\delta f_2+84\delta f_3}{462},$$

$$\phi_3=\frac{28\delta f_0+70\delta f_1+84\delta f_2+49\delta f_3}{462}$$

$$=\frac{4\delta f_0+10\delta f_1+12\delta f_2+7\delta f_3}{66}.$$

Whether any real advantage is derivable from this particular method at all commensurate with the labour and complication of the process, or whether the method itself admits of material improvement by the substitution of a different hypothesis, must for the present be left to be determined hereafter.

The former method, or that of effecting, under the best possible conditions, any prescribed small adjustments of fourth differences, is much more simple in its application, the focal values being found at once by using the coefficients contained in the first set of tables.

We may also here state, that the two practical rules given at the commencement, for the adjustment of absolute or prominent errors, are at all times useful for the verification and revision of consecutive calculations, and are indispensable for the preliminary adjustment of a series of numbers previous to interpolation, or as a preparatory for further adjustment by one of the methods just alluded to, if this be considered necessary.

Should the resulting adjustments happen to be of any considerable magnitude, the fact will indicate either that the assumed interval of the table is too great, or that the presumed approximation of the given series of numbers to the true values is not entitled to be much relied upon.

In verifying results of computation, it should be borne in mind that, though practically correct, certain slight deviations from the exact values are continually incurred by the necessity of cutting off the figures at a given place of decimals. The last digit being always put down at the nearest figure, the amount of inaccuracy from this cause can never exceed half an unit of the last digit, and, amongst the several values, it may be either in excess or defect. Hence, if three consecutive differences, according to the rule, indicate an error of only an unit in the last figure, the case may in general be passed over without suspicion of inadvertence, since a displacement to that extent may naturally be accounted for by the occasional occurrence of something like half an unit alternately in excess and defect of two consecutive numbers; but if the discrepancy of the differences should indicate a primary displacement of more than one unit, we may then be assured that an error really exists. The practical observance of this rule will obviate any unnecessary and useless re-examination of the original calculations.

It may be right also to observe generally, that the investigations that have been given in this paper have an implied reference to the elimination of casual errors or incidental imperfections, and do not necessarily apply to errors, assignable to certain specific causes, which may admit of being expressed by some mathematical function; for in such case the variations involved are no longer promiscuous in their character, and they do not contribute to produce any marked irregularities in the progression of the tabular differences. Such, for instance, are continued errors of principle in calculation, and, in some cases, personal errors of observation, which undoubtedly should be obviated or detected independently by other means, without having recourse to any theory of differences.

NOTE.—After finding the supplementary fourth difference adjustments by the formulæ (k), an analytical expression for the corrected functions may be obtained by completing the scheme of differences of the original functions (V_0, V_1, . . . V_n), that is, by treating the previous functions as zeros, and thus prefixing the values of the corresponding supplementary differences (d_{-4}, d_{-3}, d_{-2}, d_{-1}), which are simply deduced by numerical subtraction. For then, by the formulæ (ϵ), the value of an adjusted function will be

$$V_m+v_m=\Sigma'\frac{(m-\overline{x+3})(m-\overline{x+2})(m-\overline{x+1})}{2.3}(d_x+\delta d_x),$$

$$\text{or } (V+v)_m=\Sigma'\frac{(m-\overline{x+3})(m-\overline{x+2})(m-\overline{x+1})}{2.3}(d+\delta d)_x,$$

in which the ordinal characteristic (x) begins with the antecedent value -4.

Consider the case in which the fourth differences are absolutely neutralized, or in which $\delta d_0=-d_0$, $\delta d_1=-d_1$, &c. Then, in developing the terms, under Σ', of this expression, all of them, after the first four, will vanish; because the adjusted fourth differences $d_x+\delta d_x$ become severally zero when x reaches the values 0, 1, 2, &c. Therefore, in this case,

$$\begin{aligned}(V+v)_m=&\frac{(m+1)(m+2)(m+3)}{2.3}(d+\delta d)_{-4}\\&+\frac{m(m+1)(m+2)}{2.3}(d+\delta d)_{-3}\\&+\frac{(m-1)m(m+1)}{2.3}(d+\delta d)_{-2}\\&+\frac{(m-2)(m-1)m}{2.3}(d+\delta d)_{-1}\ .\ .\ .\ .\ (x);\end{aligned}$$

and this algebraic function of the third degree in m is that which represents the original series of values (V_0, V_1, . . . V_n) more approximately than any other algebraic function of the same degree.

And the most approximate algebraic function of any degree may be obtained in like manner, by employing differences of an order relatively one degree higher. The analogous geometrical problem would be the determination of that curve line of a given order which shall follow most closely upon the devious track indicated by a number of given points.

CORRESPONDENCE.

THINGS WORTH NOTING.

To the Editor of the Assurance Magazine.

SIR,—I regret to have observed of late a great falling off in that which I consider by no means the least important or least interesting department of the *Magazine*—I mean the Correspondence department. I feel sure that if those of us who are accustomed to derive information and instruction from your pages were duly to discharge the duty we owe to the *Magazine* in return, the department to which I refer would speedily show signs of renewed vitality. We all, in the course of our reading, meet from time to time with "Things worth noting," which although often apparently trivial in themselves, are yet frequently by no means destitute of interest, either from their suggestiveness, their throwing light on some point in the history of our science, their illustrating the peculiarities and comparative advantages of different methods of investigation, or the like. Now, why should we not offer, now and then, for editorial approval, if we have nothing better, a little "Budget" of such "Things" of the above or cognate character as have come under our notice? It is very conceivable that wholesome discussion might thereby not unfrequently be excited, and no small amount of instruction elicited.

The advantage to be derived by the adoption of the course I have ventured to suggest, would not, as all experience proves, be entirely, or even principally, on the side of the readers of the *Magazine*. The writers would largely participate. This is a trite point. Suffice it to say, that the information we acquire as to the amount and the completeness of our acquaintance with a subject when we try to write upon it, and the stimulus given to our endeavours to supply the deficiencies which then almost always come to light, furnish far more than a compensation for any amount of labour that the effort to put our ideas upon paper may have cost us.

"Example," we are often told, "is better than precept." I therefore propose here to exemplify, by a first instalment, what I mean by "Things worth noting."

1. The history of the celebrated formula, $a_x = vp_x(1 + a_{x+1})$, which assigns the value of an annuity upon (x) in terms of that of the corresponding benefit upon ($x+1$) is pretty well known, up to a certain point. Mr. Milne ("Introduction," pp. xv., xvi.) attributes its discovery to Thomas Simpson, who published it in 1742, and he further informs us that it was rediscovered by Euler, whose publication of it dates in 1760. Subsequent writers, as